XIII. *On a Type of Spherical Harmonics of Unrestricted Degree, Order, and Argument.*

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INTRODUCTION.

THE ordinary system of Spherical Harmonics or LAPLACE'S functions is obtained from LAPLACE'S equation

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

by choosing special values of V which satisfy this differential equation, and are of the forms

$$r^n \overset{\cos}{\underset{\sin}{\cdot}} m\phi \,.\, u_n^m(\mu), \qquad \text{or} \qquad r^{-n-1} \overset{\cos}{\underset{\sin}{\cdot}} m\phi \,.\, u_n^m(\mu),$$

where n and m are real positive integers, x, y, z being expressed in terms of r, μ, ϕ by means of the relations

$$x = r(1-\mu^2)^{\frac{1}{2}} \cos\phi, \quad y = r(1-\mu^2)^{\frac{1}{2}} \sin\phi, \quad z = r\mu;$$

the function $u_n^m(\mu)$ is a particular integral of a certain ordinary linear differential equation of the second order, and is known as LEGENDRE'S associated function of degree n and order m; these solutions, in which μ is restricted to be real and to lie between the values ± 1, and in which m is restricted to be less than or equal to n, are the solutions of LAPLACE'S equation which are required in the very important class of potential problems in which the boundary of the space considered consists of either one or two complete spheres, or of surfaces which differ only slightly from spheres.

It appears, however, that the functions $\overset{\cos}{\underset{\sin}{\cdot}} m\phi \,.\, u_n^m(\mu)$ are required for the solution of certain potential problems in which the boundaries are of forms other than complete spheres, and in some of these cases the values of n, m, and μ are not subject to the restrictions which hold in the case of the primary potential problems in which the boundaries are complete spheres. In the case in which the boundary is a spheroid or two confocal spheroids, the functions $u_n^m(\mu)$ of both kinds are required, in which, although n and m are still real integers, μ may have values which are real and greater than unity. The functions for which n is fractional or complex are required for the solution of potential problems in which the boundary consists of coaxal circular cones and of spheres with the centre at the vertex of the cones. For potential problems connected with the anchor-ring functions are required for which n is half an odd integer, and μ is greater than unity. For the space bounded by two spherical bowls with a common rim, solutions in which n is complex of the form $-\frac{1}{2}+p\iota$, and μ is greater than unity, have been applied. Solutions in which m is not an integer are sometimes of use, for example, in the potential problem for the portion of an anchor-ring cut off by two planes through the axis of the ring, which are inclined to one another at an angle not a sub-multiple of two right angles.

The expressions

$$r^n \frac{\cos}{\sin} m\phi \, . \, u_n^m(\mu), \quad r^{-n-1} \frac{\cos}{\sin} m\phi \, . \, u_n^m(\mu),$$

in which $u_n^m(\mu)$ represents any particular integral of the differential equation which it satisfies, and in which the degree n, the order m, and the argument μ may have any real or complex values, are a special type of Spherical Harmonics in the extended sense of the term, which applies to all solutions of LAPLACE's equation; the investigation of their forms reduces to that of two particular integrals, here denoted by $\mathrm{P}_n^m(\mu)$, $\mathrm{Q}_n^m(\mu)$, of the differential equation which $u_n^m(\mu)$ satisfies. The forms and properties of the functions required for various potential problems have been investigated by various writers, the investigations resting usually on a more or less independent basis; thus, for example, we possess separate theories of Toroidal functions, Conal functions, &c. It is obviously desirable that all these special functions should be treated as parts of a general theory; thus an investigation of the forms and properties of the two functions $\mathrm{P}_n^m(\mu)$, $\mathrm{Q}_n^m(\mu)$ for unrestricted values of n, m, μ is required for the consolidation of the various special results which have been obtained in connection with special potential problems. To do this by means of the modern methods applicable to linear differential equations is the object of the present memoir.

In the standard treatise of HEINE, the forms and properties of the functions $\mathrm{P}_n^m(\mu)$, $\mathrm{Q}_n^m(\mu)$ are investigated for complex values of μ, the degree n and the order m being primarily real and integral; various extensions are made to cases in which n is not so restricted, but in default of a general definition of the functions for unrestricted values of n and m, these extensions are fragmentary, incomplete, and in some cases erroneous. Many of the series which satisfy the differential equation for unrestricted values of the degree and order have been given by THOMSON and TAIT,* and a general treatment of the series has been given by OLBRICHT,† who obtains seventy-two hypergeometric functions which satisfy the differential equation, at least half of which are convergent at any given point of the μ-plane.

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* See 'Natural Philosophy,' vol. 1, Part I., Appendix B.

† See OLBRICHT, 'Studien über die Kugel- und Cylinder-functionen,' Halle, 1887.

In order that the relations between the various particular integrals in the form of series may be exhibited, it appears to be most convenient to start from integral expressions which satisfy the differential equation; this is the course adopted in the present memoir. A definition of the two functions $P_n^m(\mu)$, $Q_n^m(\mu)$ by means of integrals taken along complex paths, which shall be valid for unrestricted values of the degree and order, has been rendered possible by the introduction independently by JORDAN[*] and POCHHAMMER[†] of the use of integrals with double circuits; the use of such integrals has the great advantage over the employment of integrals taken between limits, that the constants have to satisfy no convergency conditions, and thus that the functions may be defined by means of expressions which have a definite meaning for all values of the constants.

In the special case $m = 0$, the zonal functions $P_n(\mu)$, $Q_n(\mu)$ can be completely defined by means of integrals with single circuits; this has been done by SCHLÄFLI,[‡] who bases his theory of the series which represent these functions upon such definitions.

In the first part of the present memoir the two functions $P_n^m(\mu)$, $Q_n^m(\mu)$ are defined by means of integrals in such a manner that the functions are uniform over the whole μ-plane, which, however, has a cross-cut extending along the real axis from the point $\mu = 1$ to $\mu = -\infty$; these definitions are so chosen that in the ordinary case of real integral values of n and m, the functions coincide with the well-known functions used in ordinary Spherical Harmonic Analysis; from these definitions various series are obtained which represent the functions in various domains of the μ-plane. Special conventions are made as to the meaning to be attached to the functions at points in the cross-cut. Various other integral expressions are obtained which would serve as alternative definitions of the functions. It is shown that all the known definite integral expressions for the functions in restricted cases due to LAPLACE, DIRICHLET, HEINE, and MEHLER are special cases of the more general formulæ. In the latter part of the memoir various definite integral formulæ are deduced for cases in which the degree and order are subject to special restrictions. In conclusion, the forms of the functions required for the potential problems connected with the ring, the cone, and the bowl are deduced from the general formulæ; in particular, convergent series are obtained for the tesseral toroidal functions.

As much confusion is caused by the variety of notation used by different writers, it is convenient to state here for purposes of comparison the relations between the symbols used in the most important works on the subject; for this purpose the ordinary case of integral values of n and m is the only one which has to be considered.

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[*] See 'Cours d'Analyse,' vol. 3.

[†] See various papers in volumes 35 and 36 of the 'Mathematische Annalen.'

[‡] See a tract "Ueber die beiden Heine'schen Kugelfunctionen." Bern, 1881.

HEINE uses the symbols $P_m^{(n)}(\mu)$, $\mathfrak{P}_m^{(n)}(\mu)$, $Q_m^{(n)}(\mu)$, $\mathfrak{Q}_m^{(n)}(\mu)$, which are connected with the symbols $P_n^m(\mu)$, $Q_n^m(\mu)$ used in this memoir by the relations

$$P_m^{(n)}(\mu) = P_{-m}^{(n)}(\mu) = (\mu^2 - 1)^{-\frac{1}{2}m}\,\mathfrak{P}_m^{(n)}(\mu) = \frac{1 \,.\, 2 \,.\, 3 \ldots n - m}{1 \,.\, 3 \ldots 2n - 1}\, P_n^m(\mu)$$

$$Q_m^{(n)}(\mu) = Q_{-m}^{(n)}(\mu) = (\mu^2 - 1)^{-\frac{1}{2}m}\,\mathfrak{Q}_m^{(n)}(\mu) = (-1)^m \frac{1 \,.\, 3 \ldots 2n + 1}{1 \,.\, 2 \,.\, 3 \ldots n + m}\, Q_n^m(\mu).$$

THOMSON and TAIT use the symbols $\Theta_n^{(m)}(\mu)$, $\vartheta_n^{(m)}(\mu)$, which are connected with HEINE'S $P_m^{(n)}(\mu)$ by the relations

$$(-1)^{\frac{1}{2}m}\, P_m^{(n)}(\mu) = \Theta_n^{(m)}(\mu) = \frac{2^{n-m} \,.\, n!\,(n-m)!}{(2n)!\; m!}\, \vartheta_n^{(m)}(\mu).$$

FERRERS uses $T_n^m(\mu)$ for what is denoted here by $(-1)^{\frac{1}{2}m} P_n^m(\mu)$, except in the case of a real μ lying between ± 1, in which case $T_n^m(\mu)$ and $P_n^m(\mu)$ are identical.

The Gaussian function $\Pi(x)$, which is equivalent to $\Gamma(x+1)$, is used throughout the memoir.

Definition of the functions $P_n^m(\mu)$, $Q_n^m(\mu)$ by means of definite integrals.

1. If, in the differential equation

$$(1-\mu^2)\frac{d^2V}{d\mu^2} - 2\mu\frac{dV}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\} V = 0 \quad . \quad . \quad . \quad . \quad (1),$$

which is satisfied by LEGENDRE'S associated functions, we substitute $V = (\mu^2 - 1)^{\frac{1}{2}m}\, W$, then W satisfies the differential equation

$$(1-\mu^2)\frac{d^2W}{d\mu^2} - 2(m+1)\mu\frac{dW}{d\mu} + (n-m)(n+m+1)\, W = 0 \quad . \quad . \quad (2).$$

If, in the expression on the left-hand side of (2), we substitute

$$W = \int (t^2-1)^n (t-\mu)^{-n-m-1}\, dt,$$

we find

$$\left\{(1-\mu^2)\frac{d^2}{d\mu^2} - 2(m+1)\mu\frac{d}{d\mu} + (n-m)(n+m+1)\right\}\int (t^2-1)^n (t-\mu)^{-n-m-1}\, dt$$

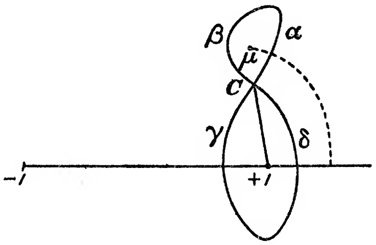
$$= -(n+m+1)\int \frac{d}{dt}\{(t^2-1)^{n+1}(t-\mu)^{-n-m-2}\}\, dt.$$

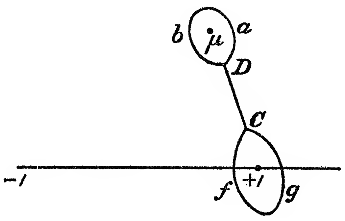
It appears thus that the differential equation (2), is satisfied by

$$W = \int (t^2 - 1)^n (t - \mu)^{-n-m-1}\, dt,$$

for unrestricted values of n and m, provided the integral is taken along a closed path, *i.e.*, one such that the integrand $(t^2 - 1)^n (t - \mu)^{-n-m-1}$ attains the same value when the path has been completely described, as that with which it commenced. The integrand has, in general, the four singular points $t = +1$, $t = -1$, $t = \mu$, $t = \infty$, and we shall see that it is possible to choose two distinct closed paths, defined with reference to these singular points, which will represent the values of W required for the two LEGENDRE'S associated functions.

2.





If the variable t, starting from a point C, describes a path in which a positive (counter-clockwise) turn is made round the point μ, then a positive turn round the point 1, then a negative turn round μ, and lastly a negative turn round 1, such a path will be closed, *i.e.*, the integrand $(t^2 - 1)^n (t - \mu)^{-n-m-1}$ will have the same value at C at the beginning and at the end of the path. In the first figure the path will be (C$\alpha\beta$C, C$\gamma\delta$C, C$\beta\alpha$C, C$\delta\gamma$C); in the second figure it will be (CD, D$ab$D, DC, C$fg$C, CD, D$ba$D, DC, C$gf$C). In POCHHAMMER'S notation, the value of V will be

$$V = (\mu^2 - 1)^{\frac{1}{2}m} \int_c^{(\mu+,\, 1+,\, \mu-,\, 1-)} (t^2 - 1)^n (t - \mu)^{-n-m-1}\, dt,$$

which will satisfy the equation (1); it is necessary to specify precisely the values of the multiple valued functions in the integral, in order that the integral may have a definite value.

First, to define the meaning of $(\mu^2 - 1)^{\frac{1}{2}m}$, let $\mu - 1 = re^{\iota\theta}$, $\mu + 1 = r'e^{\iota\theta'}$, and suppose μ to have moved from a point in the real axis for which $\mu > 1$, along any path up to its actual position; we shall suppose that $\theta = 0$, $\theta' = 0$, when μ is in the real axis and greater than unity, the value of $(\mu^2 - 1)^{\frac{1}{2}m}$ at any point will then be $(rr')^{\frac{1}{2}m} \left\{ \cos\frac{m}{2}(\theta + \theta') + \iota \sin\frac{m}{2}(\theta + \theta') \right\}$, where θ, θ' are the angles the lines joining μ and 1, μ and -1 make with the real axis; θ and θ must be restricted each to lie between $\pm \pi$, in order that a single value may be assigned to $(\mu^2 - 1)^{\frac{1}{2}m}$; by $(rr')^{\frac{1}{2}m}$ is denoted $e^{\frac{1}{2}m \log (rr')}$ where $\log (rr')$ has its real value; the value of $(\mu^2 - 1)^{\frac{1}{2}m}$ has thus been uniquely specified for all values of μ, except those which are real and lie between $+1$ and $-\infty$. Next, in $(t^2 - 1)^n = (t - 1)^n (t + 1)^n$, we shall suppose the phase of $t - 1$ to commence with the value ϕ at C, where ϕ is the angle (between $\pm \pi$) the line joining C to $+1$ makes with the positive direction of the real axis; the phase of $t + 1$ at C we shall suppose to be ϕ', where ϕ' is the angle (between $\pm \pi$) the line joining C to -1 makes with the positive direction of the real axis; if at C, $t - 1 = ke^{\iota\phi}$, $t + 1 = k'e^{\iota\phi'}$, the value of $(t^2 - 1)^n$ will be $e^{n \log (kk')} . e^{n\iota(\phi + \phi')}$ where $\log (kk')$ has its real positive value; after the positive turn round 1, $(t^2 - 1)^n$ will have become $e^{n \log (kk')} . e^{n\iota(2\pi + \phi + \phi')}$.

The phase of $t - \mu$ we shall choose to be such that it is zero when t passes through that point of the path for which $t - \mu$ is a positive real quantity, thus the initial value of $t - \mu$ at C is $\rho e^{-(\pi - \psi)\iota}$, where ψ is the angle (between $\pm \pi$) which the line Cμ makes with the positive direction of the real axis, hence $(t - \mu)^{-(n+m+1)}$ changes from $\rho^{-(n+m+1)} e^{(\pi - \psi)(n+m+1)\iota}$ to $\rho^{-(n+m+1)} e^{-(n+m+1)(\pi + \psi)\iota}$, in going from C round the point μ to C again, $\rho^{-(n+m+1)}$ denoting $e^{-(n+m+1)\log_e \rho}$, where $\log_e \rho$ has its real positive value.

3. Let us now consider the value of

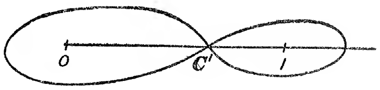
$$c_n^m (\mu^2 - 1)^{\frac{1}{2}m} \int_c^{(\mu+, 1+, \mu-, 1-)} \frac{1}{2^n} \frac{(t^2 - 1)^n}{(t - \mu)^{n+m+1}} dt \quad . \quad . \quad . \quad . \quad . \quad (3),$$

with the specifications of the phases just given, in the case in which μ is such that mod. $\frac{1}{2}(1 - \mu) < 1$. We shall make the substitution $t - 1 = (\mu - 1) u$; it will be convenient to place the path so that C is on the straight line joining 1 and μ, so that u has a real value less than unity when t is represented by the point C.

The integral becomes

$$c_n^m (\mu^2 - 1)^{\frac{1}{2}m} \int_{c'}^{(1+, 0+, 1-, 0-)} (\mu - 1)^{-m} u^n (u - 1)^{-n-m-1} \left(1 + \frac{\mu - 1}{2} u\right)^n du$$

where C′ is the point corresponding to C.



In this integral the initial phase of u at C′ is zero, that of $u - 1$ is $-\pi$, and $\left(1 + \frac{\mu - 1}{2} u\right)^n$ has the value given by the Binomial expansion.

On performing the expansion, we obtain

$$c_n^m \left(\frac{\mu + 1}{\mu - 1}\right)^{\frac{1}{2}m} \sum_{r=0}^{\infty} \frac{\Pi(n)}{\Pi(r)\, \Pi(n - r)} \left(\frac{\mu - 1}{2}\right)^r \int_{c'}^{(1+, 0+, 1-, 0-)} u^{n+r} (u - 1)^{-n-m-1} du.$$

The expression

$$e^{-\pi\iota(a+b)}\int_{C'}^{(1+,\,0+,\,1-,\,0-)} u^{a-1}(1-u)^{b-1}\,du$$

has been denoted by POCHHAMMER by $\epsilon\,(a, b)$; it has the advantage over the Eulerian integral $\int_0^1 u^{a-1}(1-u)^{b-1}\,du$ of having a definite finite value for all values of a and b. In $\epsilon\,(a, b)$, the quantity $1-u$ has the phase 0 initially at C′, so that $u-1=(1-u)\,e^{-\pi\iota}$. The principal properties of $\epsilon\,(a, b)$ are the following:—

$$(1)' \quad \epsilon\,(a, b) = \epsilon\,(b, a),$$

$$(2)' \quad \epsilon\,(a+r, b) = (-1)^r \frac{a(a+1)\ldots(a+r-1)}{(a+b)(a+b+1)\ldots(a+b+r-1)}\,\epsilon\,(a, b),$$

$$\epsilon\,(a-r, b) = (-1)^r \frac{(a+b-1)\ldots(a+b-r)}{(a-1)(a-2)\ldots(a-r)}\,\epsilon\,(a, b).$$

$$(3)' \quad \epsilon\,(a, b) = -4\sin a\pi \sin b\pi\,.\,\mathrm{E}\,(a, b)$$

when the real parts of a, b are positive, E (a, b) denoting the Eulerian integral

$$\int_0^1 u^{a-1}(1-u)^{b-1}\,du,$$

which is equal to

$$\frac{\Pi\,(a-1)\,\Pi\,(b-1)}{\Pi\,(a+b-1)}.$$

By means of (2) this theorem can be extended to the case in which the real parts of a, b are not necessarily positive.

$$(4)' \quad \epsilon\,(a, b) = \epsilon\,(1-a-b, b) = \epsilon\,(a, 1-a-b).$$

We have

$$\int^{(1+,\,0+,\,1-,\,0-)} u^{n+r}(u-1)^{-n-m-1}\,du$$

$$= e^{(n+m+1)\pi\iota}\int^{(1+,\,0+,\,1-,\,0-)} u^{n+r}(1-u)^{-n-m-1}\,du$$

$$= e^{(n+r)\pi\iota}\,\epsilon\,(n+r+1, -n-m),$$

hence, since

$$\epsilon\,(n+r+1, -n-m) = (-1)^r \frac{(n+1)(n+2)\ldots(n+r)}{(1-m)(2-m)\ldots(r-m)}\,\epsilon\,(n+1, -n-m),$$

the expression (3) becomes

$$c_n{}^m e^{n\pi\iota}\,\boldsymbol{\epsilon}\,(n+1,\;-n-m)\left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m}\Sigma\,\frac{\Pi\,(n+r)}{\Pi\,(r)\,\Pi\,(n-r)}\,\frac{1}{(1-m)\ldots(r-m)}\left(\frac{\mu-1}{2}\right)^r,$$

or

$$c_n{}^m e^{n\pi\iota}\,\boldsymbol{\epsilon}\,(n+1,\;-n-m)\left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m}\mathrm{F}\left(-n,\,n+1,\,1-m,\,\frac{1-\mu}{2}\right),$$

where F is used with the ordinary notation, for the hyper-geometric series.

In virtue of the property (4)′, we have $\boldsymbol{\epsilon}\,(n+1,\;-n-m)=\boldsymbol{\epsilon}\,(n+1,\;m)$; and from (3)′ we have

$$\boldsymbol{\epsilon}\,(n+1,\;m)=4\sin n\pi\sin m\pi\,.\,\frac{\Pi\,(n)\,\Pi\,(m-1)}{\Pi\,(n+m)}\;;$$

hence, whatever n and m may be, the expression (3) becomes

$$c_n{}^m.\,e^{n\pi\iota}.\,4\sin n\pi\sin m\pi\left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m}\frac{\Pi\,(n)\,\Pi\,(m-1)}{\Pi\,(n+m)}\,\mathrm{F}\left(-n,\,n+1,\,1-m,\,\frac{1-\mu}{2}\right).$$

4. In the case $m=0$, we have, since $\Pi\,(-m)\,\Pi\,(m-1)=\pi\operatorname{cosec} m\pi$,

$$c_n{}^0\int^{(\mu+,\,1+,\,\mu-,\,1-)}(t^2-1)^n\,(t-\mu)^{-n-1}\,dt=c_n{}^0.\,e^{n\pi\iota}.\,4\pi\sin n\pi.\,\mathrm{F}\left(-n,\,n+1,\,1,\,\frac{1-\mu}{2}\right),$$

when mod. $\frac{1}{2}(1-\mu)<1$; in accordance with usage we take the LEGENDRE'S function $\mathrm{P}_n(\mu)$ of the first kind to be given by $\mathrm{P}_n(\mu)=\mathrm{F}\left(-n,\,n+1,\,1\,\frac{1-\mu}{2}\right)$, hence, if we choose $c_n{}^0$ equal to $\dfrac{e^{-n\pi\iota}}{4\pi\sin n\pi}$, we have

$$\mathrm{P}_n(\mu)=\frac{e^{-n\pi\iota}}{4\pi\sin n\pi}\int^{(\mu+,\,1+,\,\mu-,\,1-)}\frac{1}{2^n}\,(t^2-1)^n\,(t-\mu)^{-n-1}\,dt.$$

The integral on the right-hand side defines $\mathrm{P}_n(\mu)$ over the whole plane, the function represented in the domain of the point 1 by the series, being analytically continued over the whole plane.

In order to obtain a definition of $\mathrm{P}_n{}^m(\mu)$, we shall first consider the case when m is a real positive integer, and shall then define $\mathrm{P}_n{}^m(\mu)$ for general values of m in such a way that the definition agrees with the usual definition for the special case in which m is a real integer.

When m is a positive integer, we may define $\mathrm{P}_n{}^m(\mu)$ by means of the formula $\mathrm{P}_n{}^m(\mu)=(\mu^2-1)^{\frac{1}{2}m}\,\dfrac{d^m\mathrm{P}_n(\mu)}{d\mu^m}$; thus in this case

$$\mathrm{P}_n{}^m(\mu)=\frac{e^{-n\pi\iota}}{4\pi\sin n\pi}\,\frac{\Pi\,(n+m)}{\Pi\,(n)}\,(\mu^2-1)^{\frac{1}{2}m}\int^{(\mu+,\,1+,\,\mu-,\,1-)}\frac{1}{2^n}\,(t^2-1)^n\,(t-\mu)^{-n-m-1}\,dt$$

so that in this case $c_n^m = \frac{e^{-n\pi\iota}}{4\pi\sin n\pi}\frac{\Pi(n+m)}{\Pi(n)}$. We shall choose this value of c_n^m for all values of n and m, thus obtaining a definition of the function $P_n^m(\mu)$ for all values of n and m real or complex; $P_n^m(\mu)$ is accordingly defined by the expression

$$P_n^m(\mu) = \frac{e^{-n\pi\iota}}{4\pi\sin n\pi}\,\frac{1}{2^n}\,\frac{\Pi(n+m)}{\Pi(n)}\,(\mu^2-1)^{\frac{1}{2}m}\int^{(\mu+,\,1+,\,\mu-,\,1-,)}(t^2-1)^n(t-\mu)^{-n-m-1}\,dt \quad (4)$$

for unrestricted values of n and m, the phases of the expressions in the integrand being assigned as in Art 2. In order that this function $P_n^m(\mu)$ may be a single-valued function of μ we must suppose that a cross-cut is made along the real axis from the point 1 to $-\infty$, so that the phases of $\mu - 1$, $\mu + 1$ in $(\mu^2-1)^{\frac{1}{2}m}$ are restricted to lie between $\pm\pi$, the function is then, when we take into account the remarks which we have to make in the next article, a single-valued function over the whole plane so cut, the values at points indefinitely close to one another on opposite sides of the cross-cut being in general different. It should be observed that the integrand in the integral for a given value of μ varies continuously in crossing the cross-cut which has no reference to the variable t, but applies to μ only.

When μ in such that mod. $(1-\mu) < 2$, we have

$$P_n^m(\mu) = \frac{\sin m\pi}{\pi}\,\Pi(m-1)\left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m} F\left(-n,\, n+1,\, 1-m,\, \frac{1-\mu}{2}\right)$$

$$= \frac{1}{\Pi(-m)}\left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m} F\left(-n,\, n+1,\, 1-m,\, \frac{1-\mu}{2}\right) \quad . \quad . \quad . \quad . \quad (5).$$

The formula (5) represents the function $P_n^m(\mu)$ over that part of the plane which is contained within a circle of radius 2 with its centre at the point $\mu = 1$; this function can be analytically continued over the whole plane and (with the cross-cut) the function so continued is uniform, and is given by the definite integral formula (4) which affords a general definition of the function.

When m is an integer positive or negative, the expression (4) can be simplified; in this case the integrand returns to its initial value after a positive turn round each of the points μ and 1, denoting the parts of the integral taken round $C\alpha\beta C$, $C\gamma\delta C$ (fig. 1, Art. 2) by P and Q respectively, the complete integral is

$$P + Q - Pe^{2n\pi\iota} - Qe^{(n+m+1)2\pi\iota},$$

or

$$(1 - e^{2n\pi\iota})(P + Q);$$

now $P + Q$ is the integral taken along a curve which encloses both the points μ and $+1$, and is described positively, hence, in the case in which m is an integer, the formula (4) becomes

$$\mathrm{P}_n^m(\mu) = \frac{1}{2\pi\iota}\frac{\Pi(n+m)}{\Pi(n)}\frac{(\mu^2-1)^{\frac{1}{2}m}}{2^n}\int^{(\mu+,1+)}(t^2-1)^n(t-\mu)^{-n-m-1}dt \qquad (6).$$

When $m = 0$ we have

$$\mathrm{P}_n(\mu) = \frac{1}{2\pi\iota}\int^{(\mu+,1+)}\frac{1}{2^n}(t^2-1)^n(t-\mu)^{-n-1}dt \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which agrees with the definition given by SCHLÄFLI.

The only case of failure of the formulæ (4) and (6) is that in which $n + m$ is a negative integer; in that case $\Pi(n+m)$ is infinite and the integral is zero, and the product can be evaluated by the rule for undetermined forms $0 \times \infty$; we have

$$\Pi(n+m) = -\frac{\operatorname{cosec}(m+n)\pi}{\Pi(-m-n-1)},$$

and the limiting value of

$$\frac{1}{\sin(m+n)\pi}\int^{(\mu+,1+,\mu-,1-)}(t^2-1)^n(t-\mu)^{-n-m-1}dt$$

is

$$-\frac{1}{\pi\cos(m+n)\pi}\int^{(\mu+,1+,\mu-,1-)}(t^2-1)^n(t-\mu)^{-n-m-1}\log_e(t-\mu)\,dt,$$

thus

$$\mathrm{P}_n^m(\mu) = \frac{e^{-n\pi\iota}}{4\pi\sin n\pi}$$
$$\cdot\frac{1}{2^n\pi\cos(m+n)\pi}\frac{1}{\Pi(n)\Pi(-m-n-1)}\int^{(\mu+,1+,\mu-,1-)}(t^2-1)^n(t-\mu)^{-n-m-1}\log_e(t-\mu)\,dt.$$

If in (5) we change n into $-n-1$, the hypergeometric series is unaltered, thus within the circle of convergence $\mathrm{P}_n^m(\mu)$ is equal to $\mathrm{P}_{-n-1}^m(\mu)$; it follows that the same relation holds over the whole plane; we accordingly obtain another expression for $\mathrm{P}_n^m(\mu)$ by changing n into $-n-1$ in the formula (4), we thus have

$$\mathrm{P}_n^m(\mu) = \mathrm{P}_{-n-1}^m(\mu)$$
$$= -\frac{e^{-n\pi\iota}}{4\pi\sin n\pi}\cdot 2^{n+1}\frac{\Pi(m-n-1)}{\Pi(-n-1)}(\mu^2-1)^{\frac{1}{2}m}\int^{(\mu+,1+,\mu-,1-)}(t^2-1)^{-n-1}(t-\mu)^{n-m}dt,$$
$$= -\frac{e^{-n\pi\iota}}{4\pi\sin(n-m)\pi}\cdot 2^{n+1}\frac{\Pi(n)}{\Pi(n-m)}(\mu^2-1)^{\frac{1}{2}m}\int^{(\mu+,1+,\mu-,1-)}(t^2-1)^{-n-1}(t-\mu)^{n-m}dt \qquad (8).$$

The formula (8) will serve equally with (4), as a definition of $\mathrm{P}_n^m(\mu)$; it does not appear to be easy to prove directly their equivalence.

As regards the formula (5),

$$\mathrm{P}_n^m(\mu) = \frac{1}{\Pi(-m)}\left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m} \mathrm{F}\left(-n,\, n+1,\, 1-m,\, \frac{1-\mu}{2}\right),$$

we may remark that

(α) When n is a real integer and m is not so, the series is finite, and therefore $\mathrm{P}_n^m(\mu)$ is an algebraical function.

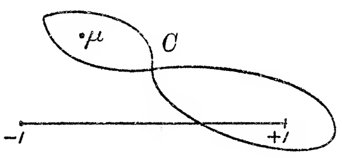
(β) When m is a real positive integer and n is not so, the formula may be written

$$\mathrm{P}_n^m(\mu) = \frac{1}{2^m}\,\frac{\Pi(n+m)}{\Pi(n-m)}\,\frac{1}{\Pi(m)}\,(\mu^2-1)^{\frac{1}{2}m}\,\mathrm{F}\left(m-n,\, n+m+1,\, m+1,\, \frac{1-\mu}{2}\right).$$

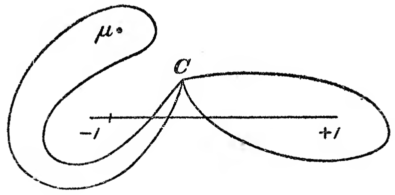
(γ) When n and m are both positive real integers, and $n > m$, it falls under case (β), the series being however finite since the first element $m - n$ of the hypergeometric series is a negative integer, thus $\mathrm{P}_n^m(\mu)$ is an algebraical function.

(δ) When n and m are both positive integers, and $n < m$, case (β) shows that $\mathrm{P}_n^m(\mu)$ is zero; in order to obtain an integral of the differential equation we must take $\Pi(n-m)\,\mathrm{P}_n^m(\mu)$ which is finite.

5.



(a.)

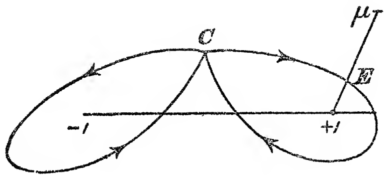


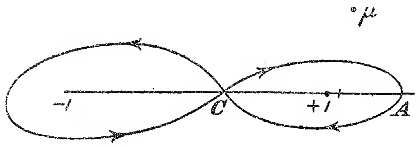
(b.)

In defining the function $\mathrm{P}_n^m(\mu)$ by means of a definite integral taken round a closed path, in which turns are made round the points μ and 1, but none round the point -1, it is necessary to specify the position of the path with reference to the point -1. The figures (a) and (b) represent two distinct paths for the same value of μ, but the integrals obtained from them will be, in general, different in value, as one path cannot be brought by continuous deformation into coincidence with the other without crossing the point -1, which is a singular point for the integrand. We shall consequently specify that the path by means of which $\mathrm{P}_n^m(\mu)$ is defined in (4), is one which does not cut the real axis between -1 and $-\infty$, or is, at all events, a path which can by continuous deformation be brought, without crossing the point -1, into a path which does not cut the real axis between -1 and $-\infty$.

6. Another closed path for the integrand $(t^2-1)^n(t-\mu)^{-n-m-1}$ is that in which

a positive turn round the point -1 is followed by a negative turn round the point $+1$.





Consider thus the expression

$$f_n^m (\mu^2 - 1)^{\frac{1}{2}m} \int_c^{(-1+,\,1-)} \frac{1}{2^n} (t^2 - 1)^n (t - \mu)^{-n-m-1}\, dt,$$

taken along the path as in either of the figures. The phase of $t - \mu$ will be measured as before; those of $t - 1$, $t + 1$, we shall take to be such that they vanish at the instant when t passes in the integration through the point A of the real axis, for which $t - 1$, $t + 1$ are both real and positive; thus, in the second figure, the initial phrases of $t - 1$, $t + 1$ at C are π and -2π respectively.

Let $t - \mu = (\mu - t)\, e^{-\iota\pi}$, then the phases of $\mu - t$ are such that at the point E, where the line joining μ and 1 cuts the path, the phase of $\mu - t$ is the angle (between $\pm \pi$) the line makes with the positive direction of the real axis; the expression becomes

$$f_n^m (\mu^2 - 1)^{\frac{1}{2}m} \int^{(-1+,\,+1-)} \frac{1}{2^n} e^{(n+m+1)\iota\pi} (t^2 - 1)^n (\mu - t)^{-n-m-1}\, dt.$$

Suppose now that mod. $\mu > 1$, the path of integration can then always be so placed that mod. t is everywhere less than mod. μ; expanding by the binomial theorem, the expression becomes

$$f_n^m (\mu^2 - 1)^{\frac{1}{2}m} \cdot \frac{1}{2^n} e^{(n+m+1)\iota\pi} \sum_{r=0}^{r=\infty} \int^{(-1+,\,+1-)} \frac{\Pi(n+m+r)}{\Pi(n+m)\,\Pi(r)} \frac{1}{\mu^{n+m+1+r}} (t^2 - 1)^n t^r\, dt.$$

To evaluate $\int^{(-1,\,1-)} (t^2 - 1)^n t^r\, dt$, we may place the path so that the two loops are exactly equal, C being half-way between the points 1 and -1; it is thus seen that the integral vanishes when r is odd, and that when r is even and equal to $2s$ it is equal to

$$-2 \int_0^{(+1+)} (t^2 - 1)^n t^{2s}\, dt;$$

making the substitution $t' = t^2$, we see that $t' - 1$, or $(t - 1)(t + 1)$ is such that its phase increases from $-\pi$ to π during the integration, we thus have

$$-\int_0^{(+1+)} (t'-1)^n \, t'^{s-\frac{1}{2}} \, dt,$$

which can easily be shown to be equal to $2\iota \sin n\pi \dfrac{\Pi(n)\,\Pi(s-\frac{1}{2})}{\Pi(n+s+\frac{1}{2})}$.

The expression with which we commenced is now reduced to the form

$$f_n^m \, . \, \frac{1}{2^n} \, . \, 2\iota \sin n\pi \, . \, e^{(n+m+1)\iota\pi} (\mu^2-1)^{\frac{1}{2}m} \sum_{s=0}^{\infty} \frac{\Pi(n+m+2s)}{\Pi(n+m)\,\Pi(2s)} \, \frac{\Pi(n)\,\Pi(s-\frac{1}{2})}{\Pi(n+s+\frac{1}{2})} \, \frac{1}{\mu^{n+m+2s+1}},$$

which is

$$f_n^m \, . \, \frac{1}{2^n} \, . \, 2\iota \sin n\pi \, . \, e^{(n+m+1)\iota\pi} (\mu^2-1)^{\frac{1}{2}m} \frac{\Pi(n)\,\Pi(-\frac{1}{2})}{\Pi(n+\frac{1}{2})}$$

$$\cdot \frac{1}{\mu^{n+m+1}} \, \mathrm{F}\left(\frac{n+m}{2}+1, \frac{n+m+1}{2}, n+\tfrac{3}{2}, \frac{1}{\mu^2}\right).$$

When n is a positive integer, we have in accordance with the usual definition of $\mathrm{Q}_n(\mu)$,

$$\mathrm{Q}_n(\mu) = \frac{1}{2^{n+1}} \cdot \frac{\Pi(-\frac{1}{2})\,\Pi(n)}{\Pi(n+\frac{1}{2})} \, \frac{1}{\mu^{n+1}} \, \mathrm{F}\left(\frac{n}{2}+1, \frac{n+1}{2}, n+\tfrac{3}{2}, \frac{1}{\mu^2}\right);$$

hence, in this case, if we take

$$f_n^0 = \frac{e^{-(n+1)\iota\pi}}{4\iota \sin n\pi}$$

we have

$$\mathrm{Q}_n(\mu) = \frac{e^{-(n+1)\iota\pi}}{4\iota \sin n\pi} \int^{(-1+,\,1-)} \frac{1}{2^n} (t^2-1)^n (t-\mu)^{-n-1} \, dt.$$

Defining $\mathrm{Q}_n^m(\mu)$ when m is a positive integer, by means of the equation

$$\mathrm{Q}_n^m(\mu) = (\mu^2-1)^{\frac{1}{2}m} \frac{d^m}{d\mu^m} \mathrm{Q}_n(\mu),$$

we have

$$\mathrm{Q}_n^m(\mu) = \frac{e^{-(n+1)\iota\pi}}{4\iota \sin n\pi} (\mu^2-1)^{\frac{1}{2}m} \frac{\Pi(n+m)}{\Pi(n)} \int^{(-1+,\,1-)} \frac{1}{2^n} (t^2-1)^n (t-\mu)^{-n-m-1} \, dt,$$

we should consequently, when m and n are positive integers, choose f_n^m equal to

$$\frac{e^{-(n+1)\iota\pi}}{4\iota \sin n\pi} \cdot \frac{\Pi(n+m)}{\Pi(n)}.$$

We shall now assign this value to f_n^m, whatever m and n are; we thus obtain the formula

$$\mathrm{Q}_n^m(\mu) = \frac{e^{-(n+1)\iota\pi}}{4\iota \sin n\pi} \cdot \frac{\Pi(n+m)}{\Pi(n)} (\mu^2-1)^{\frac{1}{2}m} \int^{(-1+,\,1-)} \frac{1}{2^n} (t^2-1)^n (t-\mu)^{-n-m-1} \, dt \quad (9),$$

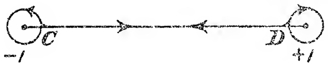
which we shall take as the definition of the function $Q_n^m(\mu)$ for unrestricted values of m and n.

When mod. $(\mu) > 1$, $Q_n^m(\mu)$ is represented by the expression

$$Q_n^m(\mu) = \frac{e^{m\iota\pi}}{2^{n+1}} \frac{\Pi(n+m)\,\Pi(-\frac{1}{2})}{\Pi(n+\frac{1}{2})} (\mu^2-1)^{\frac{1}{2}m} \frac{1}{\mu^{n+m+1}} \mathrm{F}\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}, n+\frac{3}{2}, \frac{1}{\mu^2}\right) \quad (10).$$

The uniform function obtained by continuing the function in (10) over the whole plane, with the exception of the cross-cut along the real axis from $+1$ to $-\infty$, is represented by the expression in (9).

When n is such that the real part of $n+1$ is positive, the definition (9) can be simplified, the integral being then reducible to one along a line joining the points ± 1. The path may be as in the figure; then, since the integrals along the loops round the points 1 and -1 become indefinitely small when the loops are made indefinitely small, we have



$$\int_c^{(-1+,\,1-)} (t^2-1)^n (t-\mu)^{-n-m-1}\,dt = (e^{n\pi\iota} - e^{-n\pi\iota}) \int_{-1}^{1} (1-t^2)^n (t-\mu)^{-n-m-1}\,dt$$
$$= 2\iota \sin n\pi \int_{-1}^{1} (1-t^2)^n (t-\mu)^{-n-m-1}\,dt\,;$$

hence, when $n+1$ has its real part positive, we may substitute for (9) the definition

$$Q_n^m(\mu) = \frac{e^{-(n+1)\iota\pi}}{2} \cdot \frac{\Pi(n+m)}{\Pi(n)} (\mu^2-1)^{\frac{1}{2}m} \int_{-1}^{1} \frac{1}{2^n} (1-t^2)^n (t-\mu)^{-n-m-1}\,dt$$
$$= \frac{e^{m\iota\pi}}{2^{n+1}} \frac{\Pi(n+m)}{\Pi(n)} (\mu^2-1)^{\frac{1}{2}m} \int_{-1}^{1} (1-t^2)^n (\mu-t)^{-n-m-1}\,dt. \quad . \quad . \quad . \quad (11).$$

The integral may be taken along the real axis, $(1-t^2)^n$ denoting $e^{n\log(1-t^2)}$, where the logarithm has its real value.

It will be observed that when n is a positive integer, the form (9) is undetermined $(\infty \times 0)$; we can, however, in this case use the formula (11). When n is a negative integer, the value of $Q_n^m(\mu)$, as given by (9), is in general finite, since

$$\sin n\pi\,.\,\Pi(n) = -\frac{\pi}{\Pi(-n-1)}\,;$$

if, however, $n+m$ is also a negative integer, or if m is zero, the value of $Q_n^m(\mu)$ is infinite, so that the factor $\Pi(n+m)$ must be rejected if we wish to obtain a finite solution of the differential equation.

Proof of a relation between $Q_n^m(\mu)$ and $Q_n^{-m}(\mu)$.

7. If we apply to the formula (10) the known theorem

$$F(\alpha, \beta, \gamma, x) = (1 - x)^{\gamma-\alpha-\beta} F(\gamma - \alpha, \gamma - \beta, \gamma, x)$$

we have when mod $\mu > 1$,

$$Q_n^m(\mu) = \frac{e^{m\iota\pi}}{2^{n+1}} \frac{\Pi(n+m)\Pi(-\frac{1}{2})}{\Pi(n+\frac{1}{2})} (\mu^2 - 1)^{-\frac{1}{2}m} \frac{1}{\mu^{n-m+1}} F\left(\frac{n-m+2}{2}, \frac{n-m+1}{2}, n + \tfrac{3}{2}, \frac{1}{\mu^2}\right).$$

The expression for $Q_n^{-m}(\mu)$ is obtained by writing $-m$ for m, in the formula (10); we have thus the relation

$$\frac{e^{-m\iota\pi} Q_n^m(\mu)}{\Pi(n+m)} = \frac{e^{m\iota\pi} Q_n^{-m}(\mu)}{\Pi(n-m)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

which must hold over the whole plane; it is obvious that $Q_n^{-m}(\mu)$ satisfies the differential equation (1), as that equation is unaltered by changing the sign of m. The result in (12) may also be obtained by transforming the integral in (9) by means of the transformation $(t - \mu)(t' - \mu) = \mu^2 - 1$, which is equivalent to an inversion with respect to the point μ. On making the substitution, we find

$$(\mu^2 - 1)^{\frac{1}{2}m} \int^{(-1+,\, 1-)} (t^2 - 1)^n (t - \mu)^{-n-m-1} dt$$
$$= -(\mu^2 - 1)^{-\frac{1}{2}m} \int^{(1+,\, -1-)} (t'^2 - 1)^n (t' - \mu)^{-n+m-1} dt'.$$

Corresponding to the phase $-\pi$ of $t^2 - 1$, the phase of $t'^2 - 1$ is π; also to the phase $-\pi$ of $t - \mu$, in the case in which μ is real and greater than unity, the phase of $t' - \mu$ is π, hence, in order that in the integral on the right-hand side the phases may be measured in the same way as on the left-hand side, the factor $e^{2n\pi\iota - 2(n-m+1)\pi\iota}$, or $e^{2m\pi\iota}$, must be introduced; we thus obtain

$$(\mu^2 - 1)^{\frac{1}{2}m} \int^{(-1+,\, 1-)} (t^2 - 1)^n (t - \mu)^{-n-m-1} dt$$
$$= (\mu^2 - 1)^{-\frac{1}{2}m} e^{2m\pi\iota} \int^{(-1+,\, 1-)} (t'^2 - 1)^n (t' - \mu)^{-n+m-1} dt',$$

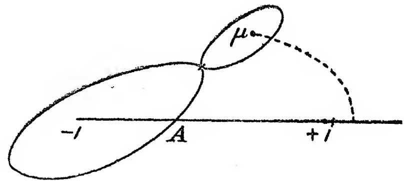
and thus the result (12) is proved.

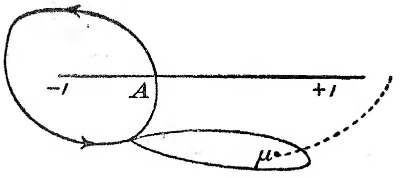
Expression for $Q_n^m(\mu)$ when mod $(\mu + 1)$ and mod $(\mu - 1)$ are less than 2.

8. It will be necessary to obtain an expression for the integral

$$(\mu^2 - 1)^{\frac{1}{2}m} \int^{(\mu+,\, -1+,\, \mu-,\, -1-)} \frac{1}{2^n} (t^2 - 1)^n (t - \mu)^{-n-m-1}\, dt,$$

analogous to the corresponding integral round the singular points μ, 1, obtained in Art. 3. To define the phases of the integrand we shall distinguish the cases in which the imaginary part of μ is positive, and is negative.





We suppose μ to move from a point in the real axis for which its value is greater than unity, up to its actual position, the path of integration being drawn as in the figures; it will be observed that as μ moves from a position on the positive side of the real axis to one on the negative side, the path cannot be displaced from its first position to the second one without crossing the singular point $+ 1$, it is therefore necessary to distinguish the two cases.

In the first figure the phase of $t - 1$ at A is $+ \pi$, and in the second figure it is $- \pi$, in both cases the phase of $t + 1$ at A is zero, and that of $t - \mu$ is measured as before.

Put $t + 1 = (\mu + 1)\, u$, the expression then becomes

$$\left(\frac{\mu - 1}{\mu + 1}\right)^{\frac{1}{2}m} \int^{(1+,\, 0+,\, 1-,\, 0-,)} u^n \left(\frac{\mu + 1}{2} u - 1\right)^n (u - 1)^{-n-m-1}\, du,$$

now we put

$$\frac{\mu + 1}{2} u - 1 = e^{\iota\pi}\left(1 - \frac{\mu + 1}{2} u\right)$$

or,

$$\frac{\mu + 1}{2} u - 1 = e^{-\iota\pi}\left(1 - \frac{\mu + 1}{2} u\right)$$

according as the imaginary part of μ is positive or negative, in both cases the phase of $1 - \frac{\mu + 1}{2} u$ is zero at A, and then $\left(1 - \frac{\mu + 1}{2} u\right)^n$ will have that value which is given by the expansion by the Binomial Theorem.

We have for the integral

$$\left(\frac{\mu - 1}{\mu + 1}\right)^{\frac{1}{2}m} e^{\pm n\pi\iota} \int^{(1+,\, 0+,\, 1-,\, 0-)} u^n \left(1 - \frac{\mu + 1}{2} u\right)^n (u - 1)^{-n-m-1}\, du,$$

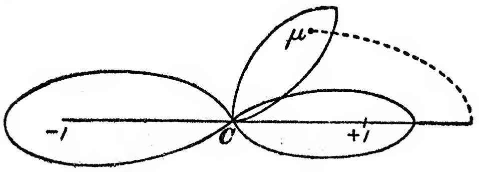
the upper or lower sign being taken in $e^{\pm n\pi\iota}$, according as μ is above or below the real axis. When mod. $(\mu + 1) < 2$, this expression can be evaluated exactly as in Art. 3, the result being obtained by writing $-\mu$ for μ; we thus find at once

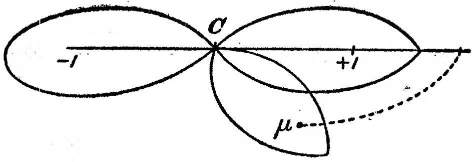
$$(\mu^2 - 1)^{\frac{1}{2}m} \int^{(\mu+,\, -1+,\, \mu-,\, -1-)} \frac{1}{2^n} (t^2 - 1)^n (t - \mu)^{-n-m-1}\, dt$$

$$= e^{2n\pi\iota} \,.\, 4 \sin n\pi \sin m\pi \frac{\Pi(n)\,\Pi(m-1)}{\Pi(n+m)} \left(\frac{\mu - 1}{\mu + 1}\right)^{\frac{1}{2}m} \mathrm{F}\left(-n,\, n+1,\, 1-m,\, \frac{1+\mu}{2}\right). \quad (13)$$

when μ is above the real axis, the exponential factor being omitted when μ is below the real axis.

9.





Let L, M, N denote the values of the integral $\int (t^2 - 1)^n (t - \mu)^{-n-m-1}\, dt$ taken along loops from C round the three points -1, 1, μ respectively, in the positive directions, the phases at C being as follows:

of $t - 1$, π in the first figure, and $-\pi$ in the second,

of $t + 1$, zero,

of $t - \mu$, $-(\pi - \phi)$, where ϕ is the (positive or negative) angle the line joining C to μ makes with the positive direction of the real axis. We have at once

$$\int_c^{(\mu+,\, 1+,\, \mu-,\, 1-)} (t^2 - 1)^n (t - \mu)^{-n-m-1}\, dt = \mathrm{N} + \mathrm{M}e^{-2\pi(m+n+1)\iota} - \mathrm{N}e^{2\pi n\iota} - \mathrm{M},$$

$$\int_c^{(\mu+,\, -1+,\, \mu-,\, -1-)} (t^2 - 1)^n (t - \mu)^{-n-m-1}\, dt = \mathrm{N} + \mathrm{L}e^{-2\pi(m+n+1)\iota} - \mathrm{N}e^{2\pi n\iota} - \mathrm{L},$$

the phases in the integrands being measured as just stated.

To express $\int_c^{(-1+,\, 1-)} (t^2 - 1)^n (t - \mu)^{-n-m-1}\, dt$, in which, as in Art. 6, the phase of $t - 1$ at C is $+\pi$, and that of $t + 1$ is -2π, we have for the value of the integral

$$\mathrm{L}e^{-2n\pi\iota} - \mathrm{M}e^{-2n\pi\iota}, \text{ or } \mathrm{L} - \mathrm{M}$$

according as μ is above or below the real axis.

It follows that

$$\int_c^{(-1+,\,1-)} (t^2-1)^n (t-\mu)^{-n-m-1}\,dt$$

$$= \frac{e^{-2n\pi\iota}}{1-e^{-2\pi(m+n)\iota}}\left\{\int_c^{(\mu+,\,1+,\,\mu-,\,1-)} (t^2-1)^n (t-\mu)^{-n-m-1}\,dt - \int_c^{(\mu+,\,-1+,\,\mu-,\,-1-)} (t^2-1)^n (t-\mu)^{-n-m-1}\,dt\right\},$$

or

$$= \frac{1}{1-e^{-2\pi(m+n)\iota}} \times \text{the same expression} \quad . \quad . \quad . \quad . \quad . \quad (14),$$

according as μ is above or below the real axis.

10. The relation (14) enables us to find the expression for $Q_n^m(\mu)$ in series, for values of μ which are such that mod. $(1+\mu)$ and mod. $(1-\mu) < 2$. Using the formulæ (5), (9), (13), we find at once

$$Q_n^m(\mu) = \frac{\pi e^{m\pi\iota}}{2\sin(m+n)\pi}\,\frac{1}{\Pi(-m)}\left\{e^{\mp n\pi\iota}\left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m} F\left(-n,\, n+1,\, 1-m,\, \frac{1-\mu}{2}\right) - \left(\frac{\mu-1}{\mu+1}\right)^{\frac{1}{2}m} F\left(-n,\, n+1,\, 1-m,\, \frac{1+\mu}{2}\right)\right\} \quad (15),$$

the upper or the lower sign being taken in $e^{\mp n\pi\iota}$, according as the imaginary part of μ is positive or negative.

When m is zero, we have

$$Q_n(\mu) = \frac{\pi}{2\sin n\pi}\left\{e^{\mp n\pi\iota}\, F\left(-n,\, n+1,\, 1,\, \frac{1-\mu}{2}\right) - F\left(-n,\, n+1,\, 1,\, \frac{1+\mu}{2}\right)\right\} \quad (16).$$

The particular case (16) agrees, when μ is above the real axis, with the result obtained by SCHLÄFLI.

If we use the relation (12) between $Q_n^m(\mu)$ and $Q_n^{-m}(\mu)$, we can write (15) in the form

$$Q_n^m(\mu) = \frac{\pi e^{m\pi\iota}}{2\sin(n-m)\pi}\,\frac{\Pi(n+m)}{\Pi(n-m)}\,\frac{1}{\Pi(m)}\left\{e^{\mp n\pi\iota}\left(\frac{\mu-1}{\mu+1}\right)^{\frac{1}{2}m} F\left(-n,\, n+1,\, 1+m,\, \frac{1-\mu}{2}\right) - \left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m} F\left(-n,\, n+1,\, 1+m,\, \frac{1+\mu}{2}\right)\right\} \quad (17).$$

When $n+m$ is a positive integer, the expression (10) shows that $Q_n^m(\mu)$ has in general a finite value, hence we see from (15), that in this case

$$e^{\mp n\pi\iota}\left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m} F\left(-n,\, n+1,\, 1-m,\, \frac{1-\mu}{2}\right) = \left(\frac{\mu-1}{\mu+1}\right)^{\frac{1}{2}m} F\left(-n,\, n+1,\, 1-m,\, \frac{1+\mu}{2}\right);$$

this result is proved by HEINE* for the special case in which n and m are both integers. We see, therefore, that when $n + m$ is a positive integer, the formula (15) is undetermined, the formula (17) must in that case be used.

When $n - m$ is a positive integer we must use (15), since (17) become in this case undetermined. When $n + m$ is a negative integer, $Q_n^m(\mu)$ is infinite, but we can take $Q_n^m(\mu)\sin(n+m)\pi$ as a finite solution of the differential equation.

When n and m are both real integers, and m is positive and $> n$, the form (17) is finite, but if $m \leqq n$ both the forms (15), (17) are undetermined, and must be modified by applying the rule for the determination of undetermined forms 0/0.

The functions $P_n^m(\mu)$, $Q_n^m(\mu)$ are defined by OLBRICHT for the general case, by means of equations, in our notation,

$$P_n^m(\mu) = \text{constant}\left(\frac{\mu-1}{\mu+1}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1+m, \frac{1-\mu}{2}\right),$$

$$Q_n^m(\mu) = \text{constant}\left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1+m, \frac{1+\mu}{2}\right),$$

this definition of $Q_n^m(\mu)$ is, however, not consistent with the usual definition as in (10), in the form of a hypergeometric series whose fourth element is $\frac{1}{\mu^2}$.

Relation between the functions Q_n^m, Q_{-n-1}^m, P_n^m.

11. In the formula (15), write $-n-1$ for n, we have then

$$Q_{-n-1}^m(\mu) = \frac{\pi e^{m\pi\iota}}{2\sin(m-n-1)\pi} \cdot \frac{1}{\Pi(-m)}\left\{e^{\pm(n+1)\pi\iota}\left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1-\mu}{2}\right)\right.$$
$$\left. - \left(\frac{\mu-1}{\mu+1}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1+\mu}{2}\right)\right\}.$$

On eliminating the second hypergeometric series between this equation and (15), we find

$$\begin{aligned} &Q_n^m(\mu)\sin(n+m)\pi - Q_{-n-1}^m(\mu)\sin(n-m)\pi \\ &\quad = \frac{\pi e^{m\pi\iota}}{2\Pi(-m)}\left(e^{-n\pi\iota} + e^{n\pi\iota}\right)\left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1-\mu}{2}\right) \\ &\quad = \pi e^{m\pi\iota}\cos m\pi \cdot P_n^m(\mu), \qquad \text{by (5).} \end{aligned}$$

We thus obtain the formula

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* See 'Kugelfunctionen,' vol. 2, pp. 238, 336.

$$\mathrm{P}_n^m(\mu) = \frac{e^{-m\pi\iota}}{\pi \cos n\pi} \{\mathrm{Q}_n^m(\mu) \sin(n+m)\pi - \mathrm{Q}_{-n-1}^m(\mu) \sin(n-m)\pi\} \quad . \quad (18).$$

This relation which has been proved to hold over the domain of the point $\mu = -1$, must hold over the whole plane.

In the case $m = 0$, we have

$$\mathrm{P}_n(\mu) = \frac{\tan n\pi}{\pi} \{\mathrm{Q}_n(\mu) - \mathrm{Q}_{-n-1}(\mu)\}.$$

If $n + m$ is a positive real integer, we have

$$\mathrm{P}_n^m(\mu) = -\frac{2}{\pi} \cdot e^{-m\pi\iota} \cos m\pi \,.\, \mathrm{Q}_{-n-1}^m(\mu).$$

If $n - m$ is a negative real integer, the relation (18) becomes

$$\mathrm{P}_n^m(\mu) = \frac{2}{\pi} e^{-m\pi\iota} \sin m\pi \,.\, \mathrm{Q}_n^m(\mu),$$

we see therefore that in this case the two functions $\mathrm{P}_n^m(\mu)$, $\mathrm{Q}_n^m(\mu)$ are not distinct.

Changing m into $-m$, in (18), we have

$$\begin{aligned}\mathrm{P}_n^{-m}(\mu)& \\ &= \frac{e^{-m\pi\iota}}{\pi\cos n\pi}\left\{\frac{\Pi(n-m)}{\Pi(n+m)}\mathrm{Q}_n^m(\mu)\sin(n-m)\pi - \frac{\Pi(-n-m-1)}{\Pi(-n+m-1)}\mathrm{Q}_{-n-1}^m(\mu)\sin(n+m)\pi\right\} \\ &= \frac{e^{-m\pi\iota}}{\pi\cos n\pi} \cdot \frac{\Pi(n-m)}{\Pi(n+m)}\sin(n-m)\pi\,\{\mathrm{Q}_n^m(\mu) - \mathrm{Q}_{-n-1}^m(\mu)\},\end{aligned}$$

hence on substituting for $\mathrm{Q}_{-n-1}^m(\mu)$ its value given by (18), we have

$$\mathrm{P}_n^{-m}(\mu) = \frac{\Pi(n-m)}{\Pi(n+m)}\left\{\mathrm{P}_n^m(\mu) - \frac{2}{\pi}e^{-m\pi\iota}\sin m\pi \,.\, \mathrm{Q}_n^m(\mu)\right\} \quad . \quad . \quad . \quad (19).$$

Remembering the relation between $\mathrm{P}_n^m(\mu)$, $\mathrm{P}_{-n-1}^m(\mu)$, we see that of the eight solutions $\mathrm{P}_n^m(\mu)$, $\mathrm{P}_{-n-1}^m(\mu)$, $\mathrm{P}_n^{-m}(\mu)$, $\mathrm{P}_{-n-1}^{-m}(\mu)$, $\mathrm{Q}_n^m(\mu)$, $\mathrm{Q}_{-n-1}^m(\mu)$, $\mathrm{Q}_n^{-m}(\mu)$, $\mathrm{Q}_{-n-1}^{-m}(\mu)$, of the equation (1), six have been expressed in terms of the other two.

Expressions for $\mathrm{P}_n^m(-\mu)$, $\mathrm{Q}_n^m(-\mu)$ *in terms of* $\mathrm{P}_n^m(\mu)$, $\mathrm{Q}_n^m(\mu)$.

12. Since the differential equation (1) is unaltered by substituting $-\mu$ for μ, it follows that $\mathrm{P}_n^m(-\mu)$, $\mathrm{Q}_n^m(-\mu)$ are particular integrals of the differential equation, and are therefore expressible in terms of $\mathrm{P}_n^m(\mu)$, $\mathrm{Q}_n^m(\mu)$.

The phases of $\mu + 1$, $\mu - 1$ in $(\mu + 1)^{\frac{1}{2}m}$, $(\mu - 1)^{\frac{1}{2}m}$ being restricted to lie between π and $-\pi$, on changing μ into $-\mu$, we must put $-\mu - 1 = e^{\mp\pi\iota}(\mu + 1)$, $-\mu + 1 = e^{\mp\pi\iota}(\mu - 1)$, where the upper or lower sign is taken according as the imaginary part of μ is positive or negative; we have therefore from (5)

$$\mathrm{P}_n^m(-\mu) = \frac{1}{\Pi(-m)}\left(\frac{\mu - 1}{\mu + 1}\right)^{\frac{1}{2}m} \mathrm{F}\left(-n,\ n + 1,\ 1 - m,\ \frac{1 + \mu}{2}\right);$$

on substituting for the series its value given by (15), we find the relation

$$\mathrm{P}_n^m(-\mu) = e^{\mp n\pi\iota}\,\mathrm{P}_n^m(\mu) - \frac{2\sin(n + m)\pi}{\pi}\,.\,e^{-m\pi\iota}\,\mathrm{Q}_n^m(\mu) \quad . \quad . \quad . \quad (20).$$

Again from (10), we have since $(-\mu)^{n+m+1} = \mu^{n+m+1}\,.\,e^{\mp(n+m+1)\iota\pi}$, where the sign is chosen as before,

$$\mathrm{Q}_n^m(-\mu) = -e^{\pm n\pi\iota}\,\mathrm{Q}_n^m(\mu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21).$$

In the particular case of a real integral value of n, we have

$$\mathrm{P}_n^m(-\mu) = (-1)^n\,\mathrm{P}_n^m(\mu) - \frac{2}{\pi}(-1)^n \sin m\pi\,.\,e^{-m\pi\iota}\,.\,\mathrm{Q}_n^m(\mu)$$

$$\mathrm{Q}_n^m(-\mu) = (-1)^{n+1}\,\mathrm{Q}_n^m(\mu).$$

Expression for $\mathrm{P}_n^m(\mu)$ *in powers of* $\frac{1}{\mu}$, *when mod* $\mu > 1$.

13. In the formula (10), the expression for $\mathrm{Q}_n^m(\mu)$ in a series of powers of $\frac{1}{\mu}$ has been obtained for the domain of $\mu = \infty$; we shall now employ the relation (18) to express $\mathrm{P}_n^m(\mu)$ in a similar manner. We find by changing n into $-n-1$ in (10),

$$\begin{aligned}\mathrm{Q}_{-n-1}^m(\mu) &\\ &= 2^n.e^{m\pi\iota}\frac{\Pi(m-n-1)\,\Pi(-\frac{1}{2})}{\Pi(-n-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\mu^{n-m}\mathrm{F}\left(\frac{m-n+1}{2},\frac{m-n}{2},\tfrac{1}{2}-n,\frac{1}{\mu^2}\right)\\ &= -2^n e^{m\pi\iota}\frac{\Pi(-\frac{1}{2})\,\Pi(n-\frac{1}{2})}{\Pi(n-m)}\frac{\cos n\pi}{\sin(n-m)\pi}(\mu^2-1)^{\frac{1}{2}m}\mu^{n-m}\mathrm{F}\left(\frac{m-n+1}{2},\frac{m-n}{2},\tfrac{1}{2}-n,\frac{1}{\mu^2}\right).\end{aligned}$$

Hence we find

$$\begin{aligned}\mathrm{P}_n^m(\mu) &\\ &= \frac{\sin(n+m)\pi}{2^{n+1}\cos n\pi}\,.\,\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})\,\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\frac{1}{\mu^{n+m+1}}\mathrm{F}\left(\frac{n+m+2}{2},\frac{n+m+1}{2},n+\tfrac{3}{2},\frac{1}{\mu^2}\right)\\ &+ 2^n\frac{\Pi(n-\frac{1}{2})}{\Pi(n-m)\,\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\mu^{n-m}\mathrm{F}\left(\frac{m-n+1}{2},\frac{m-n}{2},\tfrac{1}{2}-n,\frac{1}{\mu^2}\right) \quad . \quad . \quad . \quad (22).\end{aligned}$$

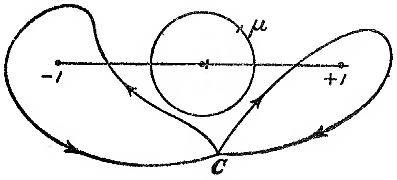
In the particular case $m = 0$, we have

$$P_n(\mu) = \frac{\tan n\pi}{2^{n+1}} \frac{\Pi(n)}{\Pi(n+\frac{1}{2})\,\Pi(-\frac{1}{2})} \frac{1}{\mu^{n+1}} F\left(\frac{n}{2}+1, \frac{n+1}{2}, n+\tfrac{3}{2}, \frac{1}{\mu^2}\right)$$
$$+ 2^n \frac{\Pi(n-\frac{1}{2})}{\Pi(n)\,\Pi(-\frac{1}{2})} \mu^n F\left(\frac{1-n}{2}, -\frac{n}{2}, \tfrac{1}{2}-n, \frac{1}{\mu^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (23).$$

It will be observed that when $n + m$ is a positive integer, the expression (20) reduces to its second term, but not so when $n + m$ is a negative integer, since $\sin(n+m)\pi \, . \, \Pi(n+m)$ is then finite. HEINE gives* as the expression for $P_n(\mu)$, when n is unrestricted, a formula which is equivalent to the second term in (23); his formula is, therefore, only correct when n is a real integer.

Expressions for P_n^m, Q_n^m in series of powers of μ, when mod. $\mu < 1$.

14. It will be convenient to obtain the expansion of $Q_n^m(\mu)$, first in powers of μ, when mod. $\mu < 1$, and afterwards to deduce the corresponding series for $P_n^m(\mu)$.



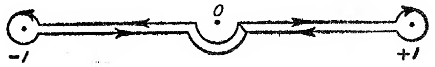
Taking the formula

$$Q_n^m(\mu) = \frac{e^{-(n+1)\iota\pi}}{4\iota \sin n\pi} (\mu^2 - 1)^{\frac{1}{2}m} \frac{\Pi(n+m)}{\Pi(n)} \int_c^{(-1+,1-)} \frac{1}{2^n} (t^2-1)^n (t-\mu)^{-n-m-1} \, dt.$$

Consider first the case in which the imaginary part of μ is positive; the path of integration can be so chosen, as in the figure, that, for every point of it, mod $t >$ mod μ; the term $(t-\mu)^{-n-m-1}$ can then be expanded in ascending powers of μ, and we thus find

$$Q_n^m(\mu) = \frac{e^{-(n+1)\iota\pi}}{4\iota \sin n\pi} (\mu^2 - 1)^{\frac{1}{2}m} \frac{1}{2^n \Pi(n)} \sum_{r=0}^{\infty} \frac{\Pi(n+m+r)}{\Pi(r)} \mu^r \int_c^{(-1+,1-)} (t^2-1)^n \, t^{-n-m-r-1} \, dt.$$

Let us now consider the integral, $\int^{(-1+,1-)} (t^2-1)^n t^p \, dt$.



First, suppose n and p to be such that the real parts of $n + 1$, $p + 1$ are both positive, the path of integration may then be as in the second figure, the loops round the points 1, -1 being indefinitely small, and the semi-circles round the point 0 being so also; the parts of the integral taken along the loops and semi-circles in the limit vanish, and we have only to consider integrals taken along the real axis. The integral consists of four parts, the following:—

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* 'Kugelfunctionen,' vol. 1, p. 38.

(1.) From 0 to -1, phase of t equal to $-\pi$, and the phases of $t-1$, $t+1$ equal to π, -2π respectively.
(2.) From -1 to 0, phase of t equal to $-\pi$, and the phases of $t-1$, $t+1$ equal to π and 0 respectively.
(3.) From 0 to 1, phases of t, $t-1$, $t+1$ equal to 0, π, 0 respectively.
(4.) From 1 to 0, phases of t, $t-1$, $t+1$ equal to 0, $-\pi$, 0 respectively.

Taking v for the modulus of t, we have in the first two parts of the integral $t = ve^{-\iota\pi}$, and in the other two parts $t = v$; hence, the integral is

$$\int_0^1 (1-v^2)^n v^p \left(e^{n\iota\pi} . e^{-2n\iota\pi} . e^{-\overline{p+1}\iota\pi} - e^{n\iota\pi} . 1 . e^{-\overline{p+1}\iota\pi} + e^{n\iota\pi} . 1 . 1 - e^{-n\iota\pi} . 1 . 1\right) dv,$$

or

$$\left(e^{n\pi\iota} - e^{-n\pi\iota}\right)\left(1 + e^{-p\pi\iota}\right) \int_0^1 (1-v^2)^n v^p \, dv,$$

which is equal to

$$2\iota \sin n\pi . \left(1 + e^{-p\pi\iota}\right) . \tfrac{1}{2} \frac{\Pi(n)\, \Pi\left(\dfrac{p-1}{2}\right)}{\Pi\left(n + \dfrac{p+1}{2}\right)} .$$

To show that the result is the same, when the real parts of $n+1$, $p+1$ are not both positive, we find by integration by parts

$$\int^{(-1+,\,1-)} (t^2-1)^n t^p \, dt = \frac{2n+p+3}{p+1} \int^{(-1+,\,1-)} (t^2-1)^n t^{p+2} \, dt,$$

and also

$$\int^{(-1+,\,1-)} (t^2-1)^n t^p \, dt = -\frac{2n+p+3}{2n+2} \int^{(-1+,\,1-)} (t^2-1)^{n+1} t^p \, dt.$$

By successive use of these two formulæ, we find

$$\int^{(-1+,\,1-)} (t^2-1)^n t^p \, dt = (-1)^\lambda \frac{\Pi\left(\dfrac{p+1}{2}+n+s\right) \Pi\left(\dfrac{p-1}{2}\right)}{\Pi\left(\dfrac{p+1}{2}+n\right) \Pi\left(\dfrac{p-1}{2}+s\right)} . \frac{\Pi\left(\dfrac{p+1}{2}+n+s+\lambda\right)}{\Pi\left(\dfrac{p+1}{2}+n+s\right)} . \frac{\Pi(n)}{\Pi(n+\lambda)}$$

$$\times \int^{(-1+,\,1-)} (t^2-1)^{n+\lambda} t^{p+2s} \, dt,$$

where λ, s are positive integers which we can so choose that the real parts of $n+\lambda+1$, $p+2s+1$ are both positive, in which case

$$\int^{(-1+,\,1-)} (t^2-1)^{n+\lambda}\, t^{p+2s}\, dt = \iota \sin(n+\lambda)\pi \,.\, (1-e^{-\overline{p+1+2s}\iota\pi}) \frac{\Pi(n+\lambda)\,\Pi\left(\frac{p+2s-1}{2}\right)}{\Pi\left(n+\lambda+\frac{p+2s+1}{2}\right)}$$

whence we find, as before,

$$\int^{(-1+,\,1-)} (t^2-1)^{n}\, t^{p}\, dt = \iota \sin n\pi \,.\, (1+e^{-p\pi\iota}) \frac{\Pi(n)\,\Pi\left(\frac{p-1}{2}\right)}{\Pi\left(n+\frac{p+1}{2}\right)}.$$

We have now, letting $p=-n-m-r-1$,

$$\mathrm{Q}_n^m(\mu) = \frac{e^{-(n+1)\iota\pi}}{4\iota\sin n\pi}(\mu^2-1)^{\frac{1}{2}m}\frac{\iota\sin n\pi}{2^n\Pi(n)}\sum_{r=0}^{\infty}(1-e^{(n+m+r)\pi\iota})\frac{\Pi(n+m+r)}{\Pi(r)}\frac{\Pi(n)\,\Pi\left(-\frac{n+m+r}{2}-1\right)}{\Pi\left(\frac{n-m-r}{2}\right)}\mu^r$$

$$= \frac{e^{-(n+1)\iota\pi}}{2^{n+2}}(1-e^{(n+m)\pi\iota})\sum_{s=0}^{\infty}\frac{\Pi\left(\frac{-n-m-2s}{2}-1\right)\Pi(n+m+2s)}{\Pi(2s)\,\Pi\left(\frac{n-m-2s}{2}\right)}\mu^{2s}$$

$$+ \frac{e^{-(n+1)\iota\pi}}{2^{n+2}}(1+e^{(n+m)\pi\iota})\sum_{s=0}^{s=\infty}\frac{\Pi\left(\frac{-n-m-2s-1}{2}-1\right)\Pi(n+m+2s+1)}{\Pi(2s+1)\,\Pi\left(\frac{n-m-2s-1}{2}\right)}\mu^{2s+1}.$$

By the known transformation theorem $\Pi(-x)\,\Pi(x-1)=\pi\operatorname{cosec} x\pi$, we have

$$\frac{\Pi\left(\frac{-n-m-2s}{2}-1\right)}{\Pi\left(\frac{n-m-2s}{2}\right)} = \frac{\Pi\left(\frac{m-n+2s}{2}-1\right)}{\Pi\left(\frac{n+m+2s}{2}\right)}\,\frac{\operatorname{cosec}\left(\frac{n+m+2s}{2}+1\right)\pi}{\operatorname{cosec}\left(\frac{m-n+2s}{2}\right)\pi}$$

$$= -\frac{\Pi\left(\frac{m-n-2}{2}+s\right)}{\Pi\left(\frac{m+n}{2}+s\right)}\,\frac{\sin\frac{m-n}{2}\pi}{\sin\frac{m+n}{2}\pi},$$

also

$$\frac{\Pi\left(\frac{-n-m-2s-1}{2}-1\right)}{\Pi\left(\frac{n-m-2s-1}{2}\right)} = \frac{\Pi\left(\frac{m-n+2s-1}{2}\right)}{\Pi\left(\frac{n+m+2s+1}{2}\right)}\,\frac{\operatorname{cosec}\left(\frac{m+n+2s+1}{2}+1\right)\pi}{\operatorname{cosec}\left(\frac{m-n+2s+1}{2}\right)\pi}$$

$$= -\frac{\Pi\left(\frac{m-n-1}{2}+s\right)}{\Pi\left(\frac{m+n+1}{2}+s\right)}\,\frac{\cos\frac{m-n}{2}\pi}{\cos\frac{m+n}{2}\pi},$$

hence the expression for $Q_n^m(\mu)$ becomes

$$Q_n^m(\mu) = -\frac{\iota e^{(m-n)\frac{\pi\iota}{2}}}{2^{n+1}}(\mu^2-1)^{\frac{1}{2}m}\sin\frac{m-n}{2}\pi \,.\, \frac{\Pi(n+m)\,\Pi\left(\frac{m-n-2}{2}\right)}{\Pi\left(\frac{n+m}{2}\right)} F\left(\frac{n+m+1}{2}, \frac{m-n}{2}, \tfrac{1}{2}, \mu^2\right)$$

$$+\frac{e^{(m-n)\frac{\pi\iota}{2}}}{2^{n+1}}(\mu^2-1)^{\frac{1}{8}m}\cos\frac{m-n}{2}\pi \,.\, \frac{\Pi(n+m+1)\,\Pi\left(\frac{m-n-1}{2}\right)}{\Pi\left(\frac{n+m+1}{2}\right)} \mu F\left(\frac{m-n+1}{2}, \frac{m+n+2}{2}, \tfrac{3}{2}, \mu^2\right) \quad (24).$$

The known transformation

$$\frac{\Pi(2x)}{\Pi(x)} = 2^{2x}\frac{\Pi(x-\frac{1}{2})}{\Pi(-\frac{1}{2})}$$

gives us

$$\frac{\Pi(n+m)}{\Pi\left(\frac{n+m}{2}\right)} = 2^{n+m}\frac{\Pi\left(\frac{n+m-1}{2}\right)}{\Pi\left(-\frac{1}{2}\right)}$$

and

$$\frac{\Pi(n+m+1)}{\Pi\left(\frac{n+m+1}{2}\right)} = 2^{n+m+1}\frac{\Pi\left(\frac{n+m}{2}\right)}{\Pi\left(-\frac{1}{2}\right)},$$

also

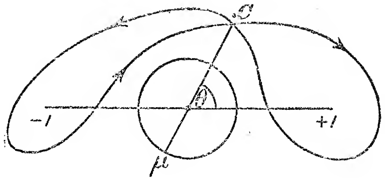
$$\Pi\left(\frac{m-n-2}{2}\right) = \frac{\pi\operatorname{cosec}\left(\frac{n+2-m}{2}\right)\pi}{\Pi\left(\frac{n-m}{2}\right)}, \quad \Pi\left(\frac{m-n-1}{2}\right) = \frac{\pi\operatorname{cosec}\left(\frac{n-m+1}{2}\right)\pi}{\Pi\left(\frac{n-m-1}{2}\right)};$$

hence the formula (24) may be written

$$Q_n^m(\mu) = -\frac{\iota e^{(m-n)\frac{\pi\iota}{2}}}{2}2^m\frac{\Pi\left(\frac{n+m+1}{2}\right)\Pi(-\frac{1}{2})}{\Pi\left(\frac{n-m}{2}\right)}(\mu^2-1)^{\frac{1}{2}m} F\left(\frac{n+m+1}{2}, \frac{m-n}{2}, \tfrac{1}{2}, \mu^2\right)$$

$$+e^{(m-n)\frac{\pi\iota}{2}}2^m\frac{\Pi\left(\frac{n+m}{2}\right)\Pi(-\frac{1}{2})}{\Pi\left(\frac{n-m-1}{2}\right)}(\mu^2-1)^{\frac{1}{2}m}\mu F\left(\frac{m-n+1}{2}, \frac{m+n+2}{2}, \tfrac{3}{2}, \mu^2\right). \quad . \quad (25).$$

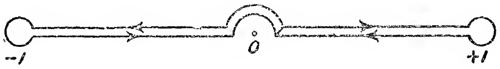
15. Next suppose the real part of μ is negative, the path of integration in the formula for $Q_n^m(\mu)$ may then be placed as in the figure, in which the line joining C and μ passes through the point $t = 0$. At C the phase of $t - \mu$ is $-(2\pi - \theta)$, and



that of t is θ, so that the phase of $1-\frac{\mu}{t}$ is -2π, thus $\left(1-\frac{\mu}{t}\right)^{-n-m-1}$ is equal to $e^{2(n+m)\pi\iota}$ times the value given by the Binomial expansion ; we have therefore

$$Q_n^m(\mu)=\frac{e^{-(n+1)\iota\pi}}{2\iota\sin n\pi}\cdot\frac{1}{2^n\,\Pi(n)}\,e^{-2n\pi\iota}\,(\mu^2-1)^{\frac{1}{2}m}\,\Sigma\,\frac{\Pi(n+m+r)}{\Pi(r)}\int_c^{(-1+,\,1-)}(t^2-1)^n\,t^{-n-m-r-1}\,dt.$$

The value of $\int_c^{(-1+,\,1-)}(t^2-1)^n\,t^p\,dt$ may be found as before by first considering the case in which the real parts of n and p are greater than unity ; in the present case,



the phases of t are π in the integral from 0 to -1, and from -1 to 0, and zero in the integrals from 0 to 1, and 1 to 0, hence the integral is equal to

$$\int_0^1(1-v^2)^n\,v^p\left(e^{n\pi\iota}\,e^{-2n\pi\iota}\,e^{\overline{p+1}\,\iota\pi}-e^{n\pi\iota}\,.\,1\,.\,e^{\overline{p+1}\,\iota\pi}+e^{n\pi\iota}\,.\,1\,.\,1-e^{-n\pi\iota}\,.\,1\,.\,1\right)dv,$$

or

$$\left(e^{n\pi\iota}-e^{-n\pi\iota}\right)\left(1+e^{p\pi\iota}\right).\,\tfrac{1}{2}\,\frac{\Pi(n)\,\Pi\left(\frac{p-1}{2}\right)}{\Pi\left((n+\frac{p+1}{2}\right)}\,.$$

The extension of this result to the case in which one or both of the quantities n, p have their real parts greater than -1, can be made as before.

We thus find after reduction, as in the preceding case,

$$Q_n^m(\mu)=\frac{\iota e^{(\frac{3}{2}m+\frac{1}{2}n)\pi\iota}}{2}\,.\,2^m\,.\,\frac{\Pi\left(\frac{n+m-1}{2}\right)\Pi(-\frac{1}{2})}{\Pi\left(\frac{n-m}{2}\right)}\,(\mu^2-1)^{\frac{1}{2}m}\,F\left(\frac{n+m+1}{2},\frac{m-n}{2},\tfrac{1}{2},\mu^2\right)$$

$$+\,e^{(\frac{3}{2}m+\frac{1}{2}n)\pi\iota}\,.\,2^m\,.\,\frac{\Pi\left(\frac{n+m}{2}\right)\Pi(-\frac{1}{2})}{\Pi\left(\frac{n-m-1}{2}\right)}\,(\mu^2-1)^{\frac{1}{2}m}\,\mu F\left(\frac{m-n+1}{2},\frac{m+n+2}{2},\tfrac{3}{2},\mu^2\right)\quad.\quad(26),$$

which is the formula that corresponds to (25).

16. In (25), change n into $-(n+1)$, we then find, after some transformation of the numerical factors,

$$Q_{-n-1}^{m}(\mu)$$

$$= -\tfrac{1}{2}e^{(m+n)\frac{\pi\iota}{2}} . 2^m \frac{\cos\frac{n+m}{2}\pi}{\sin\frac{n-m}{2}\pi} \cdot \frac{\Pi\left(\frac{m+n-1}{2}\right)\Pi\left(-\frac{1}{2}\right)}{\Pi\left(\frac{n-m}{2}\right)} (\mu^2-1)^{\frac{1}{2}m} F\left(\frac{m+n+1}{2}, \frac{m-n}{2}, \tfrac{1}{2}, \mu^2\right)$$

$$-\iota e^{(m+n)\frac{\pi\iota}{2}} . 2^m \frac{\sin\frac{n+m}{2}\pi}{\cos\frac{n-m}{2}\pi} \cdot \frac{\Pi\left(\frac{n+m}{2}\right)\Pi\left(-\frac{1}{2}\right)}{\Pi\left(\frac{n-m-1}{2}\right)} (\mu^2-1)^{\frac{1}{2}m} \mu F\left(\frac{m+n+2}{2}, \frac{m-n+1}{2}, \tfrac{3}{2}, \mu^2\right).$$

On substituting these values of $Q_n^m(\mu)$, $Q_{-n-1}^m(\mu)$ in the formula

$$P_n^m(\mu) = \frac{e^{-m\pi\iota}}{\pi\cos n\pi}\{Q_n^m(\mu)\sin(n+m)\pi - Q_{-n-1}^m(\mu)\sin(n-m)\pi\},$$

we find, after some reduction, for the case in which the imaginary part of μ is positive,

$$P_n^m(\mu) = e^{-m\pi\iota} . 2^m \cos\frac{n+m}{2}\pi . \frac{\Pi\left(\frac{n+m-1}{2}\right)}{\Pi\left(\frac{n-m}{2}\right)\Pi\left(-\frac{1}{2}\right)} (\mu^2-1)^{\frac{1}{2}m} F\left(\frac{m+n+1}{2}, \frac{m-n}{2}, \tfrac{1}{2}, \mu^2\right)$$

$$+ e^{-m\pi\iota} . 2^m \sin\frac{n+m}{2}\pi . \frac{\Pi\left(\frac{n+m}{2}\right)}{\Pi\left(\frac{n-m-1}{2}\right)\Pi\left(\frac{1}{2}\right)} (\mu^2-1)^{\frac{1}{2}m} \mu F\left(\frac{m+n+2}{2}, \frac{m-n+1}{2}, \tfrac{3}{2}, \mu^2\right) \quad (27).$$

When the imaginary part of μ is negative, we obtain in a similar manner a formula which differs from (27) only in having the exponential factor $e^{m\pi\iota}$ instead of $e^{-m\pi\iota}$.

From (27) it is seen that when $m+n$ is an integer, only one of the two hyper-geometric series is required to express $P_n^m(\mu)$, the first or the second according as $n+m$ is even or odd.

Definition of the functions P_n^m, Q_n^m for real values of μ which are less than unity.

17. The functions $P_n^m(\mu)$, $Q_n^m(\mu)$ have been defined as uniform functions of μ for all points in the plane of μ in which a cross-cut is made along the real axis from 1 to $-\infty$; at points indefinitely close to one another on opposite sides of the cross-cut the values of $P_n^m(\mu)$ or of $Q_n^m(\mu)$ will in general be different. We shall consider first

the values $P_n^m(\mu + 0 \, . \, \iota)$, $P_n^m(\mu - 0 \, . \, \iota)$ on opposite sides of the cross-cut for real values of μ lying between the values ± 1.

Referring to the expression (5), we see that in this case

$$P_n^m(\mu + 0 \, . \, \iota) = \frac{1}{\Pi(-m)} e^{-\frac{1}{2}m\pi\iota} \left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1-\mu}{2}\right)$$

$$P_n^m(\mu - 0 \, . \, \iota) = \frac{1}{\Pi(-m)} e^{\frac{1}{2}m\pi\iota} \left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1-\mu}{2}\right)$$

hence we have the relation

$$\begin{aligned} e^{\frac{1}{2}m\pi\iota} P_n^m(\mu + 0 \, . \, \iota) &= e^{-\frac{1}{2}m\pi\iota} P_n^m(\mu - 0 \, . \, \iota) \\ &= \frac{1}{\Pi(-m)} \left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1-\mu}{2}\right) \quad (28). \end{aligned}$$

It is convenient to define the function $P_n^m(\mu)$ for real values of μ between $+1$ and -1, in such a way that its value shall be real for real values of m and n; the definition which we give is that for such values of μ,

$$\begin{aligned} P_n^m(\mu) = e^{\frac{1}{2}m\pi\iota} P_n^m(\mu + 0 \, . \, \iota) &= e^{-\frac{1}{2}m\pi\iota} P_n^m(\mu - 0 \, . \, \iota) \\ &= \frac{1}{\Pi(-m)} \left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1-\mu}{2}\right) \quad . \; . \quad (29). \end{aligned}$$

From (27), we find in this case

$$\begin{aligned} P_n^m(\mu) = {} & 2^m \cos\frac{n+m}{2}\pi \, . \, \frac{\Pi\left(\frac{n+m-1}{2}\right)}{\Pi\left(\frac{n-m}{2}\right)\Pi\left(-\frac{1}{2}\right)} (1-\mu^2)^{\frac{1}{2}m} F\left(\frac{m+n+1}{2}, \frac{m-n}{2}, \frac{1}{2}, \mu^2\right) \\ & + 2^m \sin\frac{n+m}{2}\pi \, . \, \frac{\Pi\left(\frac{n+m}{2}\right)}{\Pi\left(\frac{n-m-1}{2}\right)\Pi\left(\frac{1}{2}\right)} (1-\mu^2)^{\frac{1}{2}m} F\left(\frac{m+n+2}{2}, \frac{m-n+1}{2}, \frac{3}{2}, \mu^2\right) \end{aligned}$$

when $(1-\mu^2)^{\frac{1}{2}m}$ denotes $e^{\frac{1}{2}m \log_e(1-\mu^2)}$ and $\log_e(1-\mu^2)$ has its real value.

We see from (29) that when m is zero, or an even integer, the values of the function on the opposite sides of the cross-cut are equal, so that in this case the cross-cut is necessary, so far as the function $P_n^m(\mu)$ is concerned, only from -1 to $-\infty$.

18. Next, let us consider the values of $Q_n^m(\mu)$ on opposite sides of the cross-cut for values of μ lying between ± 1; from (15) we have

$$\mathrm{Q}_n{}^m(\mu + 0\,.\,\iota) = \frac{\pi e^{m\pi\iota}}{2\sin(n+m)\pi}\,\frac{1}{\Pi(-m)}\left\{e^{-(n+\frac{1}{2}m)\pi\iota}\left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m}\mathrm{F}\left(-n,\, n+1,\, 1-m,\, \frac{1-\mu}{2}\right)\right.$$
$$\left.-e^{\frac{1}{2}m\pi\iota}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m}\mathrm{F}\left(-n,\, n+1,\, 1-m,\, \frac{1+\mu}{2}\right)\right\}$$

and

$$\mathrm{Q}_n{}^m(\mu - 0\,.\,\iota) = \frac{\pi e^{m\pi\iota}}{2\sin(n+m)\pi}\,\frac{1}{\Pi(-m)}\left\{e^{(n+\frac{1}{2}m)\pi\iota}\left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m}\mathrm{F}\left(-n,\, n+1,\, 1-m,\, \frac{1-\mu}{2}\right)\right.$$
$$\left.-e^{-\frac{1}{2}m\pi\iota}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m}\mathrm{F}\left(-n,\, n+1,\, 1-m,\, \frac{1+\mu}{2}\right)\right\},$$

from these equations we find

$$e^{-\frac{1}{2}m\pi\iota}\,\mathrm{Q}_n{}^m(\mu + 0\,.\,\iota) - e^{\frac{1}{2}m\pi\iota}\,\mathrm{Q}_n{}^m(\mu - 0\,.\,\iota)$$
$$= \frac{\pi e^{m\pi\iota}}{2\sin(n+m)\pi}\,\frac{1}{\Pi(-m)}\left(e^{-(n+m)\pi\iota} - e^{(n+m)\pi\iota}\right)\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m}\mathrm{F}\left(-n,\, n+1,\, 1-m,\, \frac{1-\mu}{2}\right),$$

hence we have the relation

$$e^{-\frac{1}{2}m\pi\iota}\,\mathrm{Q}_n{}^m(\mu + 0\,.\,\iota) - e^{\frac{1}{2}m\pi\iota}\,\mathrm{Q}_n{}^m(\mu - 0\,.\,\iota) = -\iota\pi e^{m\pi\iota}\,\mathrm{P}_n{}^m(\mu) \quad . \quad . \quad (30),$$

where $\mathrm{P}_n{}^m(\mu)$ is defined as in the last Art. In the particular case $m = 0$, (30) reduces to HEINE's relation

$$\mathrm{Q}_n(\mu + 0\,.\,\iota) - \mathrm{Q}_n(\mu - 0\,.\,\iota) = -\iota\pi\,\mathrm{P}_n(\mu).$$

It is convenient to define $\mathrm{Q}_n{}^m(\mu)$ for real values of μ between $+1$ and -1 by means of the equation

$$e^{m\pi\iota}\,.\,\mathrm{Q}_n{}^m(\mu) = \tfrac{1}{2}\left\{e^{-\frac{1}{2}m\pi\iota}\,\mathrm{Q}_n{}^m(\mu + 0\,.\,\iota) + e^{\frac{1}{2}m\pi\iota}\,\mathrm{Q}_n{}^m(\mu - 0\,.\,\iota)\right\} \quad . \quad . \quad (31),$$

which gives us

$$\mathrm{Q}_n{}^m(\mu) = \frac{\pi}{2\sin(n+m)\pi}\,\frac{1}{\Pi(-m)}\left\{\cos(n+m)\pi\,.\left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m}\mathrm{F}\left(-n,\, n+1,\, 1-m,\, \frac{1-\mu}{2}\right)\right.$$
$$\left.-\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m}\mathrm{F}\left(-n,\, n+1,\, 1-m,\, \frac{1+\mu}{2}\right)\right\}.$$

We have also

$$\mathrm{Q}_n{}^m(\mu) = -2^{m-1}\sin\frac{m+n}{2}\pi\,\frac{\Pi\left(\frac{n+m-1}{2}\right)\Pi(-\frac{1}{2})}{\Pi\left(\frac{n-m}{2}\right)}(1-\mu^2)^{\frac{m}{2}}\,\mathrm{F}\left(\frac{n+m+1}{2},\, \frac{m-n}{2},\, \tfrac{1}{2},\, \mu^2\right)$$

$$+2^{m-1}\cos\frac{m+n}{2}\,.\,\frac{\Pi\left(\frac{n+m}{2}\right)\Pi(-\frac{1}{2})}{\Pi\left(\frac{n-m-1}{2}\right)}(1-\mu^2)^{\frac{m}{2}}\,\mu\,\mathrm{F}\left(\frac{m-n+1}{2},\, \frac{m+n+2}{2},\, \tfrac{3}{2},\, \mu^2\right).$$

In the case $m = 0$, (31) agrees with HEINE's definition of the function $Q_n(\mu)$ for real values of μ between ± 1. Objections have been raised by SCHLÄFLI to this definition of $Q_n(\mu)$, on the ground that the function does not satisfy LEGENDRE's equation. There does not, however, appear to be in reality any question of principle involved; it is merely a matter of convenience to give a definition of $Q_n(\mu)$, which shall give real values of the function in the real axis, when n is real. It must, moreover, be remembered that although we have drawn the cross-cut along the real axis, it might have been drawn along any line we please joining the points ± 1, and thus the function $Q_n(\mu)$ may be regarded as satisfying the differential equation of LEGENDRE for points in or near the real axis, the surface over which the function is uniform being a different one from that which we have hitherto postulated, and the function being a linear combination of the two independent integrals of LEGENDRE's equation which we have defined and used.

19. For values of μ near that part of the real axis which is between -1 and $-\infty$, we see from the expression (10), that

$$Q_n^m(\mu + 0 \,.\, \iota)$$
$$= \frac{e^{m\pi\iota}}{2^{n+1}} \cdot \frac{\Pi(n+m)\,\Pi(-\frac{1}{2})}{\Pi(n+\frac{1}{2})} (\mu^2-1)^{\frac{1}{2}m} \,.\, e^{-(n+1)\iota\pi} \frac{1}{(-\mu)^{n+m+1}} F\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}, n+\tfrac{3}{2}, \frac{1}{\mu^2}\right),$$

$$Q_n^m(\mu - 0 \,.\, \iota)$$
$$= \frac{e^{m\pi\iota}}{2^{n+1}} \cdot \frac{\Pi(n+m)\,\Pi(-\frac{1}{2})}{\Pi(n+\frac{1}{2})} (\mu^2-1)^{\frac{1}{2}m} \,.\, e^{(n+1)\iota\pi} \,.\, \frac{1}{(-\mu)^{n+m+1}} F\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}, n+\tfrac{3}{2}, \frac{1}{\mu^2}\right),$$

where $(\mu^2-1)^{\frac{1}{2}m}$ here denotes $e^{\frac{1}{2}m \log_e(\mu^2-1)}$, the logarithm having its real positive value; we thus have

$$e^{n\pi\iota} Q_n^m(\mu + 0 \,.\, \iota) = e^{-n\pi\iota} Q_n^m(\mu - 0 \,.\, \iota) \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

and we may define $Q_n(\mu)$, for real values of μ between -1 and $-\infty$, to be equal to either of the expressions in (30) with its sign changed, thus

$$Q_n^m(\mu) = \frac{e^{m\pi\iota}}{2^{n+1}} \cdot \frac{\Pi(n+m)\,\Pi(-\frac{1}{2})}{\Pi(n+\frac{1}{2})} (\mu^2-1)^{\frac{1}{2}m} \frac{1}{(-\mu)^{n+m+1}} F\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}, n+\tfrac{3}{2}, \frac{1}{\mu^2}\right)$$

where $(\mu^2-1)^{\frac{1}{2}m}$ has the meaning given above.

To express the relation between $P_n^m(\mu)$, $P_n^m(-\mu)$, $Q_n^m(\mu)$, $Q_n^m(-\mu)$ when μ is real and lies between ± 1.

20. We have from (20), if θ lies between 0 and $\frac{1}{2}\pi$,

$$\mathrm{P}_n^m(-\cos\theta - 0\,.\,\iota) = e^{-n\pi\iota}\,\mathrm{P}_n^m(\cos\theta + 0\,.\,\iota) - \frac{2\sin(n+m)\pi}{\pi}\,e^{-m\pi\iota}\,\mathrm{Q}_n^m(\cos\theta + 0\,.\,\iota),$$

hence

$$\begin{aligned} &e^{\frac{1}{2}m\pi\iota}\,\mathrm{P}_n^m(-\cos\theta) \\ &\quad = e^{-n\pi\iota}\,.\,e^{-\frac{1}{2}m\pi\iota}\,\mathrm{P}_n^m(\cos\theta) - \frac{2\sin(n+m)\pi}{\pi}\,e^{-m\pi\iota}\,.\,e^{\frac{3}{2}m\pi\iota}\{\mathrm{Q}_n^m(\cos\theta) - \tfrac{1}{2}\iota\pi\,\mathrm{P}_n^m(\cos\theta)\},\end{aligned}$$

or

$$\mathrm{P}_n^m(-\cos\theta) = \mathrm{P}_n^m(\cos\theta)\{e^{-(n+m)\pi\iota} + \iota\sin(n+m)\pi\} - \frac{2\sin(n+m)\pi}{\pi}\,\mathrm{Q}_n^m(\cos\theta),$$

hence we have

$$\mathrm{P}_n^m(-\cos\theta) = \mathrm{P}_n^m(\cos\theta)\cos(n+m)\pi - \frac{2}{\pi}\sin(n+m)\pi\,.\,\mathrm{Q}_n^m(\cos\theta)\,. \quad (33).$$

It is easily verified by means of the formula in Arts. 17 and 18, that when $\theta = \frac{1}{2}\pi$,

$$(1 - \cos\overline{n+m}\,\pi)\,\mathrm{P}_n^m(0) = -\frac{2}{\pi}\sin(n+m)\pi\,.\,\mathrm{Q}_n^m(0),$$

hence (33) does not involve a discontinuity in the value of $\mathrm{P}_n^m(\cos\theta)$, as θ changes from 0 to π.

We have, also when θ is between 0 and $\frac{1}{2}\pi$,

$$\mathrm{Q}_n^m(-\cos\theta - 0\,.\,\iota) = -e^{n\pi\iota}\,\mathrm{Q}_n^m(\cos\theta + 0\,.\,\iota),$$

or

$$e^{\frac{m\pi}{2}}\left\{\mathrm{Q}_n^m(-\cos\theta) + \frac{\iota\pi}{2}\mathrm{P}_n^m(-\cos\theta)\right\} = -e^{n\pi\iota}\,.\,e^{\frac{3m\pi}{2}}\left\{\mathrm{Q}_n^m(\cos\theta) - \frac{\iota\pi}{2}\mathrm{P}_n^m(\cos\theta)\right\},$$

hence, by means of (33), we obtain the relation

$$\mathrm{Q}_n^m(-\cos\theta) = -\mathrm{Q}_n^m(\cos\theta)\cos(n+m)\pi - \tfrac{1}{2}\pi\sin(n+m)\pi\,.\,\mathrm{P}_n^m(\cos\theta) \quad (34).$$

When m and n are real integers we have

$$\mathrm{P}_n^m(-\cos\theta) = (-1)^{n+m}\,\mathrm{P}_n^m(\cos\theta), \quad \mathrm{Q}_n^m(-\cos\theta) = (-1)^{n+m+1}\,\mathrm{Q}_n^m(\cos\theta).$$

Expansion of $\mathrm{P}_n^m(\mu)$, $\mathrm{P}_n^m(\mu)$ *in powers of* $\mu - \sqrt{\mu^2 - 1}$.

21. If we make $(\mu - \sqrt{\mu^2 - 1})^2$, for which we shall write ξ, the independent variable in the differential equation (2), we find that the equation takes the form

$$\xi^2(1-\xi)\frac{d^2\mathrm{W}}{d\xi^2}+\xi\{\tfrac{1}{2}-m-(m+\tfrac{3}{2})\xi\}\frac{d\mathrm{W}}{d\xi}-\tfrac{1}{4}(n-m)(n+m+1)(1-\xi)\,\mathrm{W}=0.$$

Let $\mathrm{W}=\xi^{\frac{1}{2}(n+m+1)}\mathrm{W}'$; we then find, on substitution, the following differential equation for W′ :

$$\xi(1-\xi)\frac{d^2\mathrm{W}'}{d\xi^2}+\{(n+\tfrac{3}{2})-(n+2m+\tfrac{5}{2})\xi\}\frac{d\mathrm{W}'}{d\xi}-(n+m+1)(m+\tfrac{1}{2})\,\mathrm{W}'=0.$$

Comparing this with the equation,

$$\xi(1-\xi)\frac{d^2\mathrm{W}'}{d\xi^2}+\{\gamma-(\alpha+\beta+1)\xi\}\frac{d\mathrm{W}'}{d\xi}-\alpha\beta\mathrm{W}'=0,$$

which is satisfied by $\mathrm{W}'=\mathrm{F}(\alpha,\beta,\gamma,\xi)$, we see that if $\alpha=n+m+1$, $\beta=m+\frac{1}{2}$, $\gamma=n+\frac{3}{2}$, the equations are identical. It follows that our fundamental equation (1) is satisfied by

$$\mathrm{V}_1=z^{-(n+m+1)}(\mu^2-1)^{\frac{1}{2}m}\,\mathrm{F}\left(\tfrac{1}{2}+m,\ n+m+1,\ n+\tfrac{3}{2},\ \frac{1}{z^2}\right),$$

or by

$$\mathrm{V}_2=z^{n-m}(\mu^2-1)^{\frac{1}{2}m}\,\mathrm{F}\left(\tfrac{1}{2}+m,\ m-n,\ \tfrac{1}{2}-n\ \frac{1}{z^2}\right),$$

where z denotes $\mu+\sqrt{\mu^2-1}$.

In z we suppose $\sqrt{\mu^2-1}$ to be measured as hitherto, so that it has a single value at every point of the μ-plane in which a cross-cut is made along the real axis from $+1$ to $-\infty$.

It will be seen that mod z is greater than unity over the whole plane, the real part of $\sqrt{\mu^2-1}$ having the same sign as the real part of μ; on the imaginary axis z is purely imaginary.

In order to express the solutions V_1, V_2 in terms of $\mathrm{P}_n^m(\mu)$, $\mathrm{Q}_n^m(\mu)$, it will be sufficient to compare these solutions for values of μ whose modulus is very large, with the expressions (10), (22).

These latter formulæ show that for such values of μ, the principal parts of $\mathrm{Q}_n^m(\mu)$, $\mathrm{P}_n^m(\mu)$ are,

$$\frac{e^{m\pi\iota}}{2^{n+1}}\,\frac{\Pi(n+m)\,\Pi(-\frac{1}{2})}{\Pi(n+\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\mu^{-(n+m+1)},$$

$$\frac{\sin(n+m)\pi}{2^{n+1}\cos n\pi}\,\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})\,\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\mu^{-n-m-1}+2^n\frac{\Pi(n-\frac{1}{2})}{\Pi(n-m)\,\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\mu^{n-m},$$

respectively; for similar values of μ we have

$$\mathrm{V}_1=(2\mu)^{-n-m-1}(\mu^2-1)^{\frac{1}{2}m},\quad \mathrm{V}_2=(2\mu)^{n-m}(\mu^2-1)^{\frac{1}{2}m}.$$

It follows that, since V_1, V_2 must both be linear functions of $P_n^m(\mu)$, $Q_n^m(\mu)$,

$$Q_n^m(\mu)=2^m e^{m\pi\iota}\frac{\Pi(n+m)\,\Pi(-\frac{1}{2})}{\Pi(n+\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}z^{-(n+m+1)}\,F\left(\tfrac{1}{2}+m,\ n+m+1,\ n+\tfrac{3}{2},\ \frac{1}{z^2}\right) \quad (35).$$

$$P_n^m(\mu)=2^m\frac{\sin(n+m)\pi}{\cos n\pi}\,\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})\,\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}z^{-(n+m+1)}\,F\left(\tfrac{1}{2}+m,\ n+m+1,\ n+\tfrac{3}{2},\ \frac{1}{z^2}\right)$$
$$+\,2^m\frac{\Pi(n-\frac{1}{2})}{\Pi(n-m)\,\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}z^{n-m}\,F\left(\tfrac{1}{2}+m,\ m-n,\ \tfrac{1}{2}-n,\ \frac{1}{z^2}\right). \quad . \quad (36).$$

These formulæ, (35), (36), are the expressions for $Q_n^m(\mu)$, $P_n^m(\mu)$ in series of powers of $\frac{1}{z}$; the series are convergent over the whole plane.

In the particular case $m=0$, we have

$$Q_n(\mu)=\frac{\Pi(n)\,\Pi(-\frac{1}{2})}{\Pi(n+\frac{1}{2})}z^{-(n+1)}\,F\left(\tfrac{1}{2},\ n+1,\ n+\tfrac{3}{2},\ \frac{1}{z^2}\right) \quad . \quad . \quad . \quad (37).$$

$$P_n(\mu)=\tan n\pi\,.\,\frac{\Pi(n)}{\Pi(n+\frac{1}{2})\,\Pi(-\frac{1}{2})}z^{-(n+1)}\,F\left(\tfrac{1}{2},\ n+1,\ n+\tfrac{3}{2},\ \frac{1}{z^2}\right)$$
$$+\,\frac{\Pi(n-\frac{1}{2})}{\Pi(n)\,\Pi(-\frac{1}{2})}z^n\,F\left(\tfrac{1}{2},\ -n,\ \tfrac{1}{2}-n,\ \frac{1}{z^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (38).$$

The particular cases of (37), (38), in which n is a real integer, are given by HEINE.* It will be observed that the case of a real integral value of n is the only one in which $P_n(\mu)$ is represented by a single hypergeometric series. Exceptional cases of the four formulæ will be considered below.

A Second Class of Definite Integral Expressions for $P_n^m(\mu)$, $Q_n^m(\mu)$.

22. By using the definite integral forms which satisfy the hypergeometric equation, we see that the expressions

$$(\mu^2-1)^{\frac{1}{2}m}z^{-n-m-1}\int u^{n+m}(1-u)^{-\frac{1}{2}-m}\left(1-\frac{u}{z^2}\right)^{-\frac{1}{2}-m}du \quad . \quad . \quad (A)$$

$$(\mu^2-1)^{\frac{1}{2}m}z^{-n-m-1}\int u^{m-\frac{1}{2}}(1-u)^{n-m}\left(1-\frac{u}{z^2}\right)^{-n-m-1}du \quad . \quad . \quad (B)$$

satisfy the differential equation (1), the integrals being taken along closed paths, such that after a complete description of such path the integrand attains its initial value.

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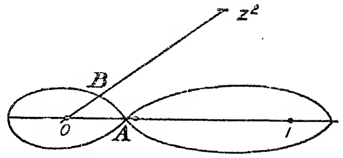
* See 'Kugelfunctionen,' vol. 1, p. 129.

In (A) or (B), n may be changed into $-n-1$, and m into $-m$; we thus have eight different forms which satisfy the differential equation, and as in each case two independent closed paths may be chosen, we obtain, on the whole, sixteen definite integrals which satisfy the differential equation (1). We shall proceed to express these definite integrals in terms of the functions $P_n^m(\mu)$, $Q_n^m(\mu)$.



Consider the integral (A), the path of integration consisting of a loop described positively round the point 1, followed by a loop positively round 0, another negatively round 1, and lastly a loop negatively round 0. When the loops are placed as in the figure, we shall suppose that the phases of u, $1-u$ initially at A are zero, and that the phase of $1-\frac{u}{z^2}$ is zero at B; when the loops are displaced into any other position the proper phases will be obtained by the principle of continuity. We have then

$$\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}\int^{(1+,\,0+,\,1-,\,0-)} u^{n+m}(1-u)^{-\frac{1}{2}-m}\left(1-\frac{u}{z^2}\right)^{-\frac{1}{2}-m} du$$

$$=\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}\sum_{r=0}^{r=\infty}\frac{\Pi(m+r-\frac{1}{2})}{\Pi(r)\,\Pi(m-\frac{1}{2})}\,\frac{1}{z^{2r}}\int^{(1+,\,0+,\,1-,\,0-)} u^{n+m+r}(1-u)^{-m-\frac{1}{2}}\,du\,;$$

now

$$\int^{(1+,\,0+,\,1-,\,0-)} u^{n+m+r}(1-u)^{-m-\frac{1}{2}}\,du = e^{(n+r+\frac{3}{2})\pi\iota}\,\epsilon\,(n+m+r+1,\,-m+\frac{1}{2})$$

$$=e^{(n+r+\frac{3}{2})\pi\iota}\cdot(-1)^r\,\frac{(n+m+1)\ldots(n+m+r)}{(n+\frac{3}{2})\ldots(n+r+\frac{1}{2})}\,\epsilon\,(n+m+1,\,-m+\frac{1}{2})$$

$$=e^{\pi(n+\frac{3}{2})\iota}\,\frac{(n+m+1)\ldots(n+m+r)}{(n+\frac{3}{2})\ldots(n+r+\frac{1}{2})}\cdot 4\pi\sin(n+m)\,\pi\,.\,\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})\,\Pi(m-\frac{1}{2})}\,.$$

Hence

$$\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}\int^{(1+,\,0+,\,1-,\,0-)} u^{n+m}(1-u)^{-\frac{1}{2}-m}\left(1-\frac{u}{z^2}\right)^{-\frac{1}{2}-m} du$$

$$=-\iota e^{\pi n\iota}\,.\,4\pi\sin(n+m)\,\pi\,.\,\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})\,\Pi(m-\frac{1}{2})}\,\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}\,F\left(n+m+1,\,m+\frac{1}{2},\,n+\frac{3}{2},\,\frac{1}{z^2}\right).$$

Comparing this result with the formula (31), we have

$$Q_n^m(\mu)$$

$$=\iota e^{(m-n)\pi\iota}\,.\,2^m\,\frac{\Pi(m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{4\pi\sin(n+m)\,\pi}\,\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}\int^{(1+,\,0+\,\,1-,\,0-)} u^{n+m}(1-u)^{-\frac{1}{2}-m}\left(1-\frac{u}{z^2}\right)^{-\frac{1}{2}-m} du \quad (39).$$

If in this expression we put $u = hz$, and make h the independent variable, we have

$$Q_n^m(\mu) = \iota e^{(m-n)\pi\iota} \,.\, 2^m \,.\, \frac{\Pi(m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{4\pi \sin(n+m)\pi} (\mu^2-1)^{\frac{1}{2}m} \int^{(\frac{1}{z}+,\, 0+,\, \frac{1}{z}-,\, 0-)} \frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}} dh \quad (40).$$

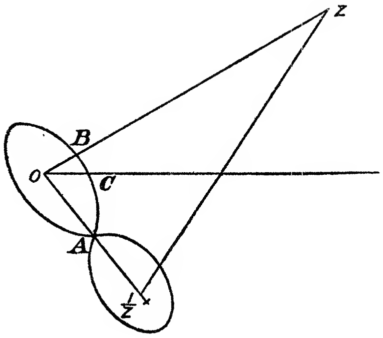
In the particular case $m = 0$, we find

$$Q_n(\mu) = \frac{\iota e^{-n\pi\iota}}{4 \sin n\pi} \int^{(\frac{1}{z}+,\, 0+,\, \frac{1}{z}-,\, 0-)} \frac{h^n}{(1-2\mu h+h^2)^{\frac{1}{2}}} dh \quad . \quad . \quad . \quad . \quad (41).$$

Using the theorem (12), we deduce from (40) the formula

$$Q_n^m(\mu) = \iota e^{(m-n)\pi\iota} \,.\, 2^{-m}$$

$$\frac{\Pi(n+m)}{\Pi(n-m)} \,.\, \frac{\Pi(-m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{4\pi \sin(n-m)\pi} (\mu^2-1)^{-\frac{1}{2}m} \int^{(\frac{1}{z}+,\, 0+,\, \frac{1}{z}-,\, 0-)} \frac{h^{n-m}}{(1-2\mu h+h^2)^{\frac{1}{2}-m}} dh \quad (42).$$



It will be observed that in the formulæ (40), (41), (42), the phases of the integrand are to be measured as follows:—Draw the figure in the h-plane corresponding to the figure we have drawn in the u-plane; the points z, $\frac{1}{z}$ correspond to the points z^2, 1 respectively; the initial phase of h at A is to be the same as that of $\frac{1}{z}$, and will therefore be zero at the first passage through C; the phase of $1 - hz$ in the product $1 - 2\mu h + h^2$, which equals $(1 - hz)\left(1 - \frac{h}{z}\right)$, will be initially zero at A, and that of $1 - \frac{h}{z}$ will be zero at B. When the figure is displaced in any manner the phases can be found from the foregoing specifications by means of the principle of continuity.

23. If the real parts of $n + m + 1$ and $\frac{1}{2} - m$ are positive, the integral in (40) can be reduced to the form

$$\left(1 - e^{(n+m)2\pi\iota}\right)\left(1 - e^{-(m+\frac{1}{2})2\pi\iota}\right) \int_0^{\frac{1}{z}} \frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}} dh,$$

thus we have

$$Q_n^m(\mu) = e^{m\pi\iota}\, 2^m \Pi\left(m - \tfrac{1}{2}\right) \Pi\left(-\tfrac{1}{2}\right) \frac{\cos m\pi}{\pi} (\mu^2 - 1)^{\frac{1}{2}m} \int_0^{\frac{1}{z}} \frac{h^{n+m}}{(1 - 2\mu h + h^2)^{m+\frac{1}{2}}}\, dh \qquad (43)$$

when the real parts of $n + m + 1$, $\frac{1}{2} - m$ are positive.

In particular

$$Q_n(\mu) = \int_0^{\frac{1}{z}} \frac{h^n}{(1 - 2\mu h + h^2)^{\frac{1}{2}}}\, dh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

provided the real part of $n + 1$ is positive.

Similarly we find from (42)

$$Q_n^m(\mu) = e^{m\pi\iota} \cdot 2^{-m} \frac{\Pi(n + m)\,\Pi\left(-\frac{1}{2}\right)}{\Pi(n - m)\,\Pi\left(m - \frac{1}{2}\right)} (\mu^2 - 1)^{-\frac{1}{2}m} \int_0^{\frac{1}{z}} \frac{h^{n-m}}{(1 - 2\mu h + h^2)^{\frac{1}{2}-m}}\, dh \quad . \quad (45)$$

provided that the real parts of $n - m + 1$, $m + \frac{1}{2}$ are positive.

In the formulæ (43), (44), (45), change h into $\frac{1}{h}$, we then find

$$Q_n^m(\mu) = e^{m\pi\iota} \cdot 2^m\, \Pi\left(m - \tfrac{1}{2}\right) \Pi\left(-\tfrac{1}{2}\right) \frac{\cos m\pi}{\pi} (\mu^2 - 1)^{\frac{1}{2}m} \int_z^{\infty} \frac{h^{m-n-1}}{(1 - 2\mu h + h^2)^{m+\frac{1}{2}}}\, dh \qquad (46)$$

when the real parts of $n + m + 1$, $\frac{1}{2} - m$ are positive.

$$Q_n(\mu) = \int_z^{\infty} \frac{h^{-n-1}}{(1 - 2\mu h + h^2)^{\frac{1}{2}}}\, dh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

where the real part of $n + 1$ is positive,

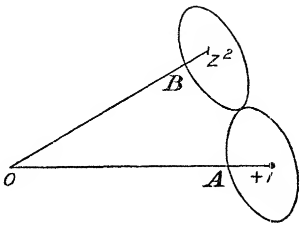
$$Q_n^m(\mu) = e^{m\pi\iota} 2^{-m} \frac{\Pi(n + m)\,\Pi\left(-\frac{1}{2}\right)}{\Pi(n - m)\,\Pi\left(m - \frac{1}{2}\right)} (\mu^2 - 1)^{-\frac{1}{2}m} \int_z^{\infty} \frac{h^{-n-m-1}}{(1 - 2\mu h + h^2)^{\frac{1}{2}-m}}\, dh \quad . \quad (48),$$

when the real parts of $n - m + 1$, $m + \frac{1}{2}$, are positive.

24. Next, consider the expression

$$(\mu^2 - 1)^{\frac{1}{2}m} z^{-n-m-1} \int^{(1+,\, z^2-)} u^{n+m} (1 - u)^{-\frac{1}{2}-m} \left(1 - \frac{u}{z^2}\right)^{-\frac{1}{2}-m} du.$$

Suppose that the phases of u, $1 - u$ are zero, when the point A in which the path cuts the real axis between 0 and 1 is reached, and that $1 - \frac{u}{z^2}$ has its phase zero at the point B in which $\frac{u}{z^2}$ is real and less than unity.



Transform the integral by means of the equation $u = z^2 - (z^2 - 1)v$, so that v is the independent variable, we have then,

$$-(\mu^2 - 1)^{\frac{1}{2}m} z^{-n-m-1} \int^{(1+,\,0-)} z^{2(n+m)} \left\{1 - \left(1 - \frac{1}{z^2}\right) v\right\}^{n+m} (z^2 - 1)^{-\frac{1}{2}-m}$$

$$(v - 1)^{-\frac{1}{2}-m} \left(1 - \frac{1}{z^2}\right)^{-\frac{1}{2}-m} v^{-\frac{1}{2}-m} (z^2 - 1)\, dv,$$

or,

$$-(\mu^2 - 1)^{\frac{1}{2}m} z^{n+3m} (z^2 - 1)^{-2m} \int^{(1+,\,0-)} \left\{1 - \left(1 - \frac{1}{z^2}\right) v\right\}^{n+m} (v - 1)^{-\frac{1}{2}-m} v^{-\frac{1}{2}-m}\, dv\,;$$

in this integral $v - 1$ has the phase zero at that point of the path in which v is positive and greater than unity; this expression may be written in the form

$$-\frac{1}{2^{2m}} (\mu^2 - 1)^{-\frac{1}{2}m} z^{n+m} \Sigma (-1)^r \left(1 - \frac{1}{z^2}\right)^r \frac{\Pi(n+m)}{\Pi(n+m-r)\,\Pi(r)} \int^{(1+,\,0-)} v^{-\frac{1}{2}-m+r} (v - 1)^{-\frac{1}{2}-m}\, dv\,;$$

now

$$\int^{(1+,\,0-)} v^{-\frac{1}{2}-m+r} (v - 1)^{-\frac{1}{2}-m}\, dv = 2\iota \cos m\pi \,.\, \frac{\Pi(-\frac{1}{2} - m)\,\Pi(-\frac{1}{2} - m + r)}{\Pi(-2m + r)},$$

hence the expression becomes

$$-\frac{1}{2^{2m}} (\mu^2 - 1)^{-\frac{1}{2}m} z^{n+m} \frac{\Pi(-\frac{1}{2} - m)\,\Pi(-\frac{1}{2} - m)}{\Pi(-2m)}\, 2\iota \cos m\pi \mathrm{F}\left(-n - m, \tfrac{1}{2} - m, 1 - 2m, 1 - \frac{1}{z^2}\right),$$

or

$$-\frac{1}{2^{2m}} (\mu^2 - 1)^{-\frac{1}{2}m} \,.\, z^{n+m} \,.\, \frac{\Pi(2m - 1)}{\{\Pi(m - \frac{1}{2})\}^2} \,.\, 4\pi\iota \sin m\pi \mathrm{F}\left(-n - m, \tfrac{1}{2} - m, 1 - 2m, 1 - \frac{1}{z^2}\right),$$

or

$$-(\mu^2 - 1)^{-\frac{1}{2}m} z^{n+m} 2\iota \sin m\pi \,.\, \frac{\Pi(m - 1)\,\Pi(-\frac{1}{2})}{\Pi(m - \frac{1}{2})} \mathrm{F}\left(-n - m, \tfrac{1}{2} - m, 1 - 2m, 1 - \frac{1}{z^2}\right).$$

If we use the known transformation

$$\mathrm{F}(\alpha, \beta, \gamma, x) = \frac{\Pi(\gamma-\alpha-\beta-1)\,\Pi(\gamma-1)}{\Pi(\gamma-\alpha-1)\,\Pi(\gamma-\beta-1)}\,\mathrm{F}(\alpha, \beta, 1+\alpha+\beta-\gamma, 1-x)$$
$$+\frac{\Pi(\alpha+\beta-\gamma-1)\,\Pi(\gamma-1)}{\Pi(\alpha-1)\,\Pi(\beta-1)}\,(1-x)^{\gamma-\alpha-\beta}\,\mathrm{F}(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1, x),$$

we find for our expression

$$-2\iota\sin m\pi\,.\,(\mu^2-1)^{-\frac{1}{2}m}z^{n+m}\frac{\Pi(m-1)\,\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})}\left\{\frac{\Pi(n-\frac{1}{2})\,\Pi(-2m)}{\Pi(n-m)\,\Pi(-m-\frac{1}{2})}\right.$$
$$\mathrm{F}\left(-n-m, \tfrac{1}{2}-m, \tfrac{1}{2}-n, \frac{1}{z^2}\right)+\frac{\Pi(-n-\frac{3}{2})\,\Pi(-2m)}{\Pi(-n-m-1)\,\Pi(-\frac{1}{2}-m)}\,z^{-2n-1}$$
$$\left.\mathrm{F}\left(n-m+1, -m+\tfrac{1}{2}, \tfrac{3}{2}+n, \frac{1}{z^2}\right)\right\},$$

which can be written

$$-\frac{\pi\iota}{2^{2m-1}}(\mu^2-1)^{-\frac{1}{2}m}\frac{1}{\Pi(m-\frac{1}{2})}\left\{\frac{\Pi(n-\frac{1}{2})}{\Pi(n-m)}z^{n+m}\,\mathrm{F}\left(-n-m, \tfrac{1}{2}-m, \tfrac{1}{2}-n, \frac{1}{z^2}\right)\right.$$
$$\left.+\frac{\sin(n+m)\pi}{\cos n\pi}\,\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})}\,z^{m-n-1}\,\mathrm{F}\left(n+1-m, \tfrac{1}{2}-m, n+\tfrac{3}{2}, \frac{1}{z^2}\right)\right\}$$

or, if we use the transformation

$$\mathrm{F}(\alpha, \beta, \gamma, x) = (1-x)^{\gamma-\alpha-\beta}\,\mathrm{F}(\gamma-\alpha, \gamma-\beta, \gamma, x),$$

it can be written

$$-\frac{2\pi\iota}{\Pi(m-\frac{1}{2})}\left\{\frac{\Pi(n-\frac{1}{2})}{\Pi(n-m)}z^{n-m}(\mu^2-1)^{\frac{1}{2}m}\,\mathrm{F}\left(\tfrac{1}{2}+m, m-n, \tfrac{1}{2}-n, \frac{1}{z^2}\right)\right.$$
$$\left.+\frac{\sin(n+m)\pi}{\cos n\pi}\,\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})}\,z^{-m-n-1}(\mu^2-1)^{\frac{1}{2}m}\,\mathrm{F}\left(\tfrac{1}{2}+m, n+m+1, n+\tfrac{3}{2}, \frac{1}{z^2}\right)\right\};$$

on referring to (22), we see that it is equal to

$$-\frac{2\pi\iota\,\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})}\,\frac{1}{2^m}\,\mathrm{P}_n^{\,m}(\mu);$$

we thus obtain the formula

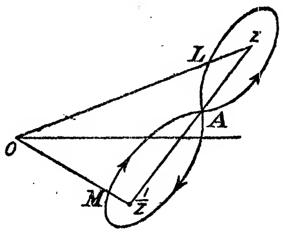
$$\mathrm{P}_n^{\,m}(\mu)$$
$$=\frac{\iota}{2\pi}\,2^m\,\frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})}\,z^{-n-m-1}(\mu^2-1)^{\frac{1}{2}m}\int^{(1+,\,z^2-)}u^{n+m}(1-u)^{-\frac{1}{2}-m}\left(1-\frac{u}{z^2}\right)^{-\frac{1}{2}-m}du \quad (49);$$

on making the substitution $u = hz$, this becomes

$$\mathrm{P}_n^m(\mu) = \frac{\iota}{2\pi} 2^m \frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})} (\mu^2 - 1)^{\frac{1}{2}m} \int^{(\frac{1}{z}+,\, z-)} \frac{h^{n+m}}{(1 - 2\mu h + h^2)^{m+\frac{1}{2}}} dh,$$

or

$$\mathrm{P}_n^m(\mu) = \frac{1}{2\pi\iota} \cdot 2^m \frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})} (\mu^2 - 1)^{\frac{1}{2}m} \int^{(z+,\, \frac{1}{z}-)} \frac{h^{n+m}}{(1 - 2\mu h + h^2)^{m+\frac{1}{2}}} dh \quad . \quad (50).$$



In this integral the phases of $1 - hz$, $1 - \frac{h}{z}$ are to be zero at the points M, L in which the lines joining the points $\frac{1}{z}$, z to the origin cut the path.

In the particular case $m = 0$, we have

$$\mathrm{P}_n(\mu) = \frac{1}{2\pi\iota} \int^{(z+,\, \frac{1}{z}-)} \frac{h^n}{(1 - 2\mu h + h^2)^{\frac{1}{2}}} dh,$$

which is reducible to

$$\mathrm{P}_n(\mu) = \frac{1}{2\pi\iota} \int \frac{h^n}{(1 - 2\mu h + h^2)^{\frac{1}{2}}} dh \quad . \; . \; . \; . \; . \; . \; . \; . \quad (51),$$

the integral being taken along a closed path which includes both the points z, $\frac{1}{z}$, and excludes the point 0.

By changing n into $-(n + 1)$, we obtain the formulæ

$$\mathrm{P}_n^m(\mu) = \frac{1}{2\pi\iota} 2^m \frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})} (\mu^2 - 1)^{\frac{1}{2}m} \int^{(z+,\, \frac{1}{z}-)} \frac{h^{m-n-1}}{(1 - 2\mu h + h^2)^{m+\frac{1}{2}}} dh \quad . \quad (52),$$

$$\mathrm{P}_n(\mu) = \frac{1}{2\pi\iota} \int \frac{h^{-n-1}}{(1 - 2\mu h + h^2)^{\frac{1}{2}}} dh \quad . \; . \; . \; . \; . \; . \; . \; . \quad (53),$$

the integral (53) being taken as in (51).

25. Next consider the expression (B), the path being the same as in Art. 22; we find

$$\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}\int^{(1+,\,0+,\,1-,\,0-)} u^{m-\frac{1}{2}}(1-u)^{n-m}\left(1-\frac{u}{z^2}\right)^{-n-m-1}du$$

$$=\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}\Sigma\frac{\Pi(n+m+r)}{\Pi(r)\Pi(n+m)}\frac{1}{z^{2r}}\int^{(1+,\,0+,\,1-,\,0-)} u^{r+m-\frac{1}{2}}(1-u)^{n-m}du.$$

Now

$$\int^{(1+,\,0+,\,1-,\,0-)} u^{r+m-\frac{1}{2}}(1-u)^{n-m}du = e^{(n+r+\frac{3}{2})\iota\pi}\epsilon\left(r+m+\tfrac{1}{2},\, n-m+1\right)$$

$$=e^{(n+r+\frac{3}{2})\iota\pi}\,.\,(-1)^r\frac{(m+\frac{1}{2})\dots(m+r-\frac{1}{2})}{(n+\frac{3}{2})\dots(n+r+\frac{1}{2})}\epsilon\left(m+\tfrac{1}{2},\, n-m+1\right)$$

$$=e^{(n+\frac{3}{2})\iota\pi}\frac{(m+\frac{1}{2})\dots(m+r-\frac{1}{2})}{(n+\frac{3}{2})\dots(n+r+\frac{1}{2})}\,.\,4\cos m\pi\sin(n-m)\pi\frac{\Pi(m-\frac{1}{2})\Pi(n-m)}{\Pi(n+\frac{1}{2})},$$

hence the expression becomes

$$e^{(n+\frac{3}{2})\iota\pi}\,.\,4\cos m\pi\sin(n-m)\pi$$

$$.\,\frac{\Pi(m-\frac{1}{2})\Pi(n-m)}{\Pi(n+\frac{1}{2})}\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}\,.\,F\left(n+m+1,\, m+\tfrac{1}{2},\, n+\tfrac{3}{2},\, \frac{1}{z^2}\right).$$

Comparing this with the expression (31), for $Q_n^m(\mu)$, we see that

$$Q_n^m(\mu)=\iota e^{(m-n)\iota\pi}\frac{2^m}{4\cos m\pi\sin(n-m)\pi}\frac{\Pi(n+m)}{\Pi(n-m)}\frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})}\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}$$

$$.\int^{(1+,\,0+,\,1-,\,0-)} u^{m-\frac{1}{2}}(1-u)^{n-m}\left(1-\frac{u}{z^2}\right)^{-n-m-1}du\quad. \qquad (54).$$

26. We shall now consider the expression

$$\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}\int^{(1+,\,z^2+,\,1-,\,z^2-)} u^{m-\frac{1}{2}}(1-u)^{n-m}\left(1-\frac{u}{z^2}\right)^{-n-m-1}du.$$

Using the transformation $u=z^2-(z^2-1)v$, the expression becomes

$$-(\mu^2-1)^{-\frac{1}{2}m}\frac{z^{n+m}}{2^{2m}}\int^{(1+,\,z^2+,\,1-,\,z^2-)}\left\{1-\frac{z^2-1}{z^2}v\right\}^{m-\frac{1}{2}}(v-1)^{n-m}v^{-n-m-1}dv,$$

which can be expanded into the form

$$-(\mu^2-1)^{-\frac{1}{2}m}\frac{z^{n+m}}{2^{2m}}\Sigma\frac{\Pi(m-\frac{1}{2})}{\Pi(r)\Pi(m-r-\frac{1}{2})}(-1)^r\left(1-\frac{1}{z^2}\right)^r\int^{(1+,\,z^2+,\,1-,\,z^2-)}(v-1)^{n-m}v^{r-n-m-1}dv.$$

Now

$$\int^{(1+,\ z^2+,\ 1-,\ z^2-)} (v-1)^{n-m}\, v^{r-n-m-1}\, dv = e^{(r-2m+1)\pi\iota}\, \epsilon\,(n-m+1\ \ r-n-m)$$

$$= -e^{(r-2m)\pi\iota}\,.\,(-1)^r\,.\,\frac{(-n-m)(1-n-m)\ldots(r-n-m-1)}{(1-2m)(2-2m)\ldots(r-2m)}\,\epsilon\,(n-m+1,\ -n-m)$$

$$= e^{-2m\pi\iota}\,\frac{(-n-m)\ldots(r-n-m-1)}{(1-2m)\ldots(r-2m)}\,.\,4\pi\sin(n-m)\pi\sin(n+m)\pi\,\frac{\Pi(n-m)\,\Pi(-n-m-1)}{\Pi(-2m)}$$

$$= -e^{-2m\pi\iota}\,.\,2^{2m}\,.\,2\pi\sin 2m\pi\sin(n-m)\pi$$

$$.\,\frac{\Pi(n-m)}{\Pi(n+m)}\,\frac{\Pi(m-\frac{1}{2})\,\Pi(m-1)}{\Pi(-\frac{1}{2})}\,.\,\frac{(-n-m)\ldots(r-n-m-1)}{(1-2m)\ldots(r-2m)}\,.$$

Thus the expression becomes

$$e^{-2m\pi\iota}\,.\,2\pi\sin 2m\pi\sin(n-m)\pi$$

$$\frac{\Pi(n-m)}{\Pi(n+m)}\,\frac{\Pi(m-\frac{1}{2})\,\Pi(m-1)}{\Pi(-\frac{1}{2})}\,z^{n+m}\,(\mu^2-1)^{-\frac{1}{2}m}\,\mathrm{F}\left(\tfrac{1}{2}-m,\ -n-m,\ 1-2m,\ 1-\frac{1}{z^2}\right).$$

As in Art. 21, this expression can be shown to be equal to

$$e^{-2m\pi\iota}\,.\,4\pi^2\cos m\pi\,.\,\sin(n-m)\pi\,.\,\frac{\Pi(n-m)}{\Pi(n+m)}\,\frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})}\,\frac{1}{2^m}\,\mathrm{P}_n^{\,m}(\mu),$$

hence we have the formula

$$\mathrm{P}_n^{\,m}(\mu) = \frac{2^m e^{2m\pi\iota}}{4\pi^2}\,\frac{1}{\cos m\pi\sin(n-m)\pi}\,\frac{\Pi(n+m)}{\Pi(n-m)}\,\frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})}\,\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}$$

$$\int^{(1+,\ z^2+,\ 1-,\ z^2-)} u^{m-\frac{1}{2}}\,(1-u)^{n-m}\left(1-\frac{n}{z^2}\right)^{-n-m-1} du\ .\quad (55),$$

and in particular when $m = 0$

$$\mathrm{P}_n(\mu) = \frac{1}{4\pi^2\sin n\pi}\,\frac{1}{z^{n+1}}\int^{(1+,\ z^2+,\ 1-,\ z^2-)} u^{-\frac{1}{2}}\,(1-u)^n\left(1-\frac{u}{z^2}\right)^{-n-1} du\ .\quad (56).$$

A third class of definite integrals which represent the function $\mathrm{P}_n^{\,m}(\mu)$, $\mathrm{Q}_n^{\,m}(\mu)$.

27. If we put $\mu^2 = \mu'$, we find that the differential equation (2) becomes, when μ' is made the independent variable,

$$\mu'(1-\mu')\frac{d^2\mathrm{W}}{d\mu'^2} + \left(\tfrac{1}{2} - \frac{2m+3}{2}\,\mu'\right)\frac{d\mathrm{W}}{d\mu'} + \frac{(n-m)(n+m+1)}{4}\,\mathrm{W} = 0.$$

We see that this equation is the differential equation satisfied by the hypergeometric series $F(\alpha, \beta, \gamma, \mu')$, where $\alpha = \frac{m-n}{2}$, $\beta = \frac{m+n+1}{2}$, $\gamma = \frac{1}{2}$; we thus see that the differential equation (1) is satisfied by either of the expressions

$$(\mu^2-1)^{\frac{1}{2}m}\int u^{\frac{n+m-1}{2}}(1-u)^{\frac{-m-n+1}{2}}(1-\mu^2u)^{\frac{n-m}{2}}\,du,$$

$$(\mu^2-1)^{\frac{1}{2}m}\int u^{\frac{m-n-2}{2}}(1-u)^{\frac{n-m}{2}}(1-\mu^2u)^{-\frac{n+m+1}{2}}\,du,$$

when, as in the other cases, the integrals are taken along closed paths. We thus obtain a third class of definite integrals, by which the functions $P_n^m(\mu)$, $Q_n^m(\mu)$ can be represented. It is unnecessary to obtain the exact expressions for the functions in four of these definite integral expressions, as all the results of interest may be obtained from the two classes which have been already considered.

The existence of these three classes of definite integrals which satisfy the fundamental differential equation (1) is equivalent to the result obtained by OLBRICHT, that the equation is satisfied by three distinct RIEMANN'S P-functions,

$$P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \frac{1}{2}m & -n & \frac{1}{2}m & \frac{1-\mu}{2} \\ -\frac{1}{2}m & n+1 & -\frac{1}{2}m & \end{array}\right\},$$

$$P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ -\frac{n}{2} & m & -\frac{n}{2} & \frac{\mu+\sqrt{\mu^2-1}}{2\sqrt{\mu^2-1}} \\ \frac{n+1}{2} & -m & \frac{n+1}{2} & \end{array}\right\},$$

$$P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ -\frac{n}{2} & \frac{m}{2} & 0 & \frac{1}{1-\mu^2} \\ \frac{n+1}{2} & -\frac{m}{2} & \frac{1}{2} & \end{array}\right\}.$$

Expansion of $P_n^m(\mu)$, $Q_n^m(\mu)$ *in powers of* $\frac{\mu\pm\sqrt{\mu^2-1}}{2\sqrt{\mu^2-1}}$.

28. In the formula

$$Q_n^m(\mu) = \iota e^{(m-n)\pi\iota}\,.\,2^m\,.\,\frac{\Pi(m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{4\pi\sin(n+m)\pi}(\mu^2-1)^{\frac{1}{2}m}\int^{(\frac{1}{z}+,\,0+,\,\frac{1}{z}-,\,0-)}\frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}}\,dh \quad . \quad (40)$$

change from h to w as independent variable in the integral, where $h = \frac{1}{z}(1 - w)$; we then have

$$\begin{aligned} Q_n^m(\mu) = & - \iota e^{(m-n)\pi\iota} \, . \, 2^m \, . \, \frac{\Pi(m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{4\pi \sin(n+m)\pi} \\ & \cdot \frac{z^{m-n}}{(z^2-1)^{m+\frac{1}{2}}} (\mu^2-1)^{\frac{1}{2}m} \int^{(0+,\,1+,\,0-,\,1-)} u^{-m-\frac{1}{2}+r} (1-u)^{n+m} \left(1 + \frac{u}{z^2-1}\right)^{-m-\frac{1}{2}} du \\ = & - \iota e^{(m-n)\pi\iota} \, . \, 2^m \, . \, \frac{\Pi(m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{4\pi \sin(n+m)\pi} \\ & \cdot \frac{z^{m-n}}{(z^2-1)^{m+\frac{1}{2}}} (\mu^2-1)^{\frac{1}{2}m} \sum_{r=0} (-1)^r . \frac{\Pi(m+\frac{1}{2}+r)}{\Pi(r)\,\Pi(m-\frac{1}{2})} \frac{1}{(z^2-1)^r} \int^{(0+,\,1+,\,0-,\,1-)} u^{-m-\frac{1}{2}+r} (1-u)^{n+m} du. \end{aligned}$$

On evaluating the definite integrals we find

$$\begin{aligned} & Q_n^m(\mu) \\ & = e^{m\pi\iota} . 2^m . \frac{\Pi(-\frac{1}{2})\,\Pi(n+m)}{\Pi(n+\frac{1}{2})} \frac{z^{m-n}}{(z^2-1)^{m+\frac{1}{2}}} (\mu^2-1)^{\frac{1}{2}m} \, F\left(m+\tfrac{1}{2}, -m+\tfrac{1}{2}, n+\tfrac{3}{2}, \frac{1}{1-z^2}\right) \quad (57), \end{aligned}$$

which gives an expression for $Q_n^m(\mu)$ in powers of $\frac{\mu - \sqrt{\mu^2 - 1}}{2\sqrt{\mu^2 - 1}}$, which is convergent for the part of the plane over which this expression has its modulus less than unity.

Using the formula

$$P_n^m(\mu) = \frac{e^{-m\pi\iota}}{\pi \cos n\pi} \{ Q_n^m(\mu) \sin(n + m)\pi - Q_{-n-1}^m(\mu) \sin(n - m)\pi \},$$

we find from (57)

$$\begin{aligned} & P_n^m(\mu) \\ & = \frac{2^m \Pi(-\frac{1}{2})}{\pi} \left\{ \frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})} \frac{\sin(n+m)\pi}{\cos n\pi} \frac{z^{m-n}}{(z^2-1)^{m+\frac{1}{2}}} (\mu^2-1)^{\frac{1}{2}m} \, F\left(m+\tfrac{1}{2}, -m+\tfrac{1}{2}, n+\tfrac{3}{2}, \frac{1}{1-z^2}\right) \right. \\ & \qquad \left. + \frac{\Pi(n-\frac{1}{2})}{\Pi(n-m)} \frac{z^{m+n+1}}{(z^2-1)^{m+\frac{1}{2}}} (\mu^2-1)^{\frac{1}{2}m} \, F\left(m+\tfrac{1}{2}, -m+\tfrac{1}{2}, -n+\tfrac{1}{2}, \frac{1}{1-z^2}\right) \right\} . \end{aligned}$$

Now by the known formula for the transformation of a hypergeometric series whose fourth element is $1 - x$, into a linear function of series whose fourth element is x, we find

$$\begin{aligned} & F\left(m + \tfrac{1}{2}, - m + \tfrac{1}{2}, n + \tfrac{3}{2}, - \frac{z^2}{1 - z^2}\right) \\ & = \frac{\Pi(n - \frac{1}{2})\,\Pi(n + \frac{1}{2})}{\Pi(n - m)\,\Pi(n + m)} F\left(m + \tfrac{1}{2}, - m + \tfrac{1}{2}, \tfrac{1}{2} - n, \frac{1}{1 - z^2}\right) \\ & + \frac{\Pi(- n - \frac{3}{2})\,\Pi(n + \frac{1}{2})}{\Pi(m - \frac{1}{2})\,\Pi(- m - \frac{1}{2})} \frac{1}{(1 - z^2)^{n+\frac{1}{2}}} \left(- \frac{z^2}{1 - z^2}\right)^{-n-\frac{1}{2}} F\left(\tfrac{1}{2} + m, \tfrac{1}{2} - m, n + \tfrac{3}{2}, \frac{1}{1 - z^2}\right), \end{aligned}$$

and thence, after some reduction

$$\mathrm{P}_n^m(\mu) = \frac{2^m\,\Pi(-\tfrac{1}{2})}{\pi}\,\frac{\Pi(n+m)}{\Pi(n+\tfrac{1}{2})}\left\{e^{-(m-\frac{1}{2})\pi\iota}\frac{z^{m-n}}{(z^2-1)^{m+\frac{1}{2}}}(\mu^2-1)^{\frac{1}{2}m}\,\mathrm{F}\left(\tfrac{1}{2}+m,\tfrac{1}{2}-m,n+\tfrac{3}{2},\frac{1}{1-z^2}\right)\right.$$

$$\left.+\frac{z^{m+n+1}}{(z^2-1)^{m+\frac{1}{2}}}(\mu^2-1)^{\frac{1}{2}m}\,\mathrm{F}\left(\tfrac{1}{2}+m,\tfrac{1}{2}-m,n+\tfrac{3}{2},\frac{-z^2}{1-z^2}\right)\right\}. \quad (58).$$

This formula expresses $\mathrm{P}_n^m(\mu)$ in powers of $\dfrac{\mu\pm\sqrt{\mu^2-1}}{2\sqrt{\mu^2-1}}$.

29. Let $\mu=\cos\theta$, then remembering that $\mathrm{P}_n^m(\cos\theta)=e^{\frac{1}{2}m\pi\iota}\,\mathrm{P}_n^m(\cos\theta+0\,.\,\iota)$, we have

$$\mathrm{P}_n^m(\cos\theta)$$

$$=\frac{2^m\,\Pi(-\tfrac{1}{2})}{\pi}\,\frac{\Pi(n+m)}{\Pi(n+\tfrac{1}{2})}\,e^{m\pi\iota}\sin^m\theta\left\{e^{-(m-\frac{1}{2})\pi\iota}\frac{e^{-(n+\frac{1}{2})\iota\theta}}{(2e^{\frac{1}{2}\iota\pi}\sin\theta)^{m+\frac{1}{2}}}\,\mathrm{F}\left(\tfrac{1}{2}+m,\tfrac{1}{2}-m,n+\tfrac{3}{2},\frac{-e^{-\iota\theta}}{2e^{\iota\pi/2}\sin\theta}\right)\right.$$

$$\left.+\frac{e^{(n+\frac{1}{2})\iota\theta}}{(2e^{\iota\pi/2}\sin\theta)^{m+\frac{1}{2}}}\,\mathrm{F}\left(\tfrac{1}{2}+m,\tfrac{1}{2}-m,n+\tfrac{3}{2},\frac{e^{\iota\theta}}{2e^{\iota\pi/2}\sin\theta}\right)\right\}.$$

Hence

$$\mathrm{P}_n^m(\cos\theta)=\frac{2}{\sqrt{\pi}}\,\frac{\Pi(n+m)}{\Pi(n+\tfrac{1}{2})}\left[\frac{\cos\left(\overline{n+\tfrac{1}{2}}\,\theta-\dfrac{\pi}{4}+\dfrac{m\pi}{2}\right)}{(2\sin\theta)^{\frac{1}{2}}}+\frac{1^2-4m^2}{2\,.\,2n+3}\,\frac{\cos\left(\overline{n+\tfrac{3}{2}}\,\theta-\dfrac{3\pi}{4}+\dfrac{m\pi}{2}\right)}{(2\sin\theta)^{\frac{3}{2}}}\right.$$

$$\left.+\frac{1^2-4m^2\,.\,3^2-4m^2}{2\,.\,4\,.\,2n+3\,.\,2n+5}\,\frac{\cos\left(\overline{n+\tfrac{5}{2}}\,\theta-\dfrac{5\pi}{4}+\dfrac{m\pi}{2}\right)}{(2\sin\theta)^{\frac{5}{2}}}+\ldots\right].\,(59);$$

this series represents $\mathrm{P}_n^m(\cos\theta)$ for unrestricted values of n and m, provided it is convergent, which is the case when $\dfrac{\pi}{6}<\theta<\dfrac{5\pi}{6}$.

To find the corresponding expression for $\mathrm{Q}_n^m(\cos\theta)$, we have from (57),

$$\mathrm{Q}_n^m(\cos\theta+0\,.\,\iota)$$

$$=e^{m\pi\iota}\,.\,2^m\,.\,\frac{\Pi(-\tfrac{1}{2})\,\Pi(n+m)}{\Pi(n+\tfrac{1}{2})}\;\frac{e^{(m-n)\iota\theta}\,(e^{\frac{1}{2}\iota\pi}\sin\theta)^m}{e^{(m+\frac{1}{2})\iota\theta}\,(2e^{\iota\pi/2}\sin\theta)^{m+\frac{1}{2}}}\,\mathrm{F}\left(\tfrac{1}{2}+m,\tfrac{1}{2}-m,n+\tfrac{3}{2},\frac{-e^{-\iota\theta}}{2e^{\iota\pi/2}\sin\theta}\right)$$

$$=e^{m\pi\iota}\,\frac{\Pi(-\tfrac{1}{2})\,\Pi(n+m)}{\Pi(n+\tfrac{1}{2})}\;\frac{e^{-(m+\frac{1}{2})\theta\iota-\iota\pi/4}}{(2\sin\theta)^{\frac{1}{2}}}\left\{1-\frac{1^2-4m^2}{2\,.\,2n+3}\,\frac{e^{-\iota(\theta+\pi/2)}}{2\sin\theta}\right.$$

$$\left.+\frac{1^2-4m^2\,.\,3^2-4m^2}{2\,.\,4\,.\,2n+3\,.\,2n+5}\,\frac{e^{-2\iota(\theta+\pi/2)}}{2\sin\theta}-\ldots\right\}$$

Similarly we find

$$\mathrm{Q}_n^m(\cos\theta-0\,.\,\iota)=e^{m\pi\iota}\,.\,\frac{\Pi(-\tfrac{1}{2})\,\Pi(n+m)}{\Pi(n+\tfrac{1}{2})}\,.\,\frac{e^{\iota(n+\frac{1}{2})\theta+\iota\pi/4}}{(2\sin\theta)^{\frac{1}{2}}}\left\{1-\frac{1^2-4m^2}{2\,.\,2n+3}\,.\,\frac{e^{\iota(\theta+\pi/2)}}{2\sin\theta}+\,..\right\}$$

thence using the relation

$$e^{m\pi\iota}\,\mathrm{Q}_n^m(\cos\theta) = \tfrac{1}{2}\left\{e^{-\frac{1}{2}m\pi\iota}\,\mathrm{Q}_n^m(\mu + 0\,.\,\iota) + e^{\frac{1}{2}m\pi\iota}\,\mathrm{Q}_n^m(\mu - 0\,.\,\iota)\right\} \quad . \quad . \quad (29)$$

we find

$$\mathrm{Q}_n^m(\cos\theta) = \sqrt{\pi}\,.\,\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})}\left\{\frac{\cos\left(\overline{n+\frac{1}{2}}\,\theta + \frac{\pi}{4} + \frac{m\pi}{2}\right)}{(2\sin\theta)^{\frac{1}{2}}} - \frac{1^2-4m^2}{2\,.\,2n+3}\,.\,\frac{\cos\left(\overline{n+\frac{3}{2}}\,\theta + \frac{3\pi}{4} + \frac{m\pi}{2}\right)}{(2\sin\theta)^{\frac{3}{2}}}\right.$$

$$\left. + \frac{1^2-4m^2\,.\,3^2-4m^2}{2\,.\,4\,.\,2n+3\,.\,2n+5}\,\frac{\cos\left(\overline{n+\frac{5}{2}}\,\theta + \frac{5\pi}{4} + \frac{m\pi}{2}\right)}{(2\sin\theta)^{\frac{5}{2}}} - \ldots\right\} \quad (60),$$

the convergency condition for this series is the same as for (59).

It may be remarked that the series (57) is convergent if μ is a real positive quantity greater than unity, ($= \cosh\psi$) provided $\psi > \frac{1}{2}\log 2$, or $\cosh\psi > \frac{3}{2\sqrt{2}}$; in that case we have

$$\mathrm{Q}_n^m(\cosh\psi) = e^{m\pi\iota}\,\sqrt{\pi}\,\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})}\,\frac{e^{-(n+\frac{1}{2})\psi}}{(2\sinh\psi)^{\frac{1}{2}}}\left\{1 - \frac{1^2-4m^2}{2\,.\,2n+3}\,\frac{e^{-\psi}}{2\sinh\psi}\right.$$

$$\left. + \frac{1^2-4m^2\,.\,3^2-4m^2}{2\,.\,4\,.\,2n+3\,.\,2n+5}\,.\,\frac{e^{-2\psi}}{(2\sinh\psi)^{\frac{1}{2}}} - \ldots\right\} \quad (61),$$

where $\cosh\psi > \frac{3}{2\sqrt{2}}$.

The corresponding series for $\mathrm{P}_n^m(\cosh\psi)$ is not convergent.

30. The series (59), (60) are convergent, provided θ lies between $\frac{\pi}{6}$ and $\frac{5\pi}{6}$; it will now however be shown that in case m and n are real, and $n + m - 1$, $\frac{1}{2} + m$ are positive, a finite number of terms of the series will represent approximately the values of $\mathrm{P}_n^m(\cos\theta)$, $\mathrm{Q}_n^m(\cos\theta)$ when the restriction as to the value of θ is removed. To prove this, it will be necessary to estimate the remainder after any number of terms in the series (57).

It has been shown by DARBOUX* that if x is a complex quantity, MACLAURIN'S theorem takes the form

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^r}{r!}\,.\,\lambda\,.\,f^r(\theta' x)$$

where θ' is a proper fraction, and λ denotes some quantity whose modulus is not greater than unity. Applying this result to the expansion in Art. 28, by which (57) was obtained, we see that the remainder after r terms of the series for $\mathrm{Q}_n^m(\cos\theta + 0\,.\,\iota)$ is

$$-\iota\,.\,e^{m\pi\iota}\,.\,\frac{\Pi(m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{4\pi\sin(m+n)\pi}\,e^{-(n+\frac{1}{2})\iota\theta-\frac{\iota\pi}{4}}\,.\,\frac{1}{\sqrt{2\sin\theta}}\,.\,\frac{1}{(z^2-1)^r}\,(-1)^r\,\frac{\Pi(m+\frac{1}{2}+r)}{\Pi(r)\,\Pi(m-\frac{1}{2})}$$

$$\int^{(0+,\,1+,\,0-,\,1-)} u^{-m-\frac{1}{2}+r}\,(1-u)^{n+m}\,\lambda\left(1 + \frac{\theta' u}{z^2-1}\right)^{-m-\frac{1}{2}}du,$$

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* See LIOUVILLE'S 'Journal,' Series III., vol. 4.

where λ is a quantity whose modulus is less than unity; suppose r so great that $r-m+\frac{1}{2}$ is positive, the integral may then be replaced by

$$-4\sin\left(m-\tfrac{1}{2}+r\right)\pi\sin(n+m)\pi\int_0^1 u^{-m-\frac{1}{2}+r}(1-u)^{n+m}\lambda\left(1+\frac{\theta' u}{z^2-1}\right)^{-m-\frac{1}{2}}du$$

where the integral is now taken along the real axis. We have now

$$1+\frac{\theta' u}{z^2-1}=1-\frac{\iota\theta' u\,.\,e^{-\iota\theta}}{2\sin\theta}=1-\frac{\theta' u}{2}-\iota\frac{\theta' u\cot\theta}{2},$$

the modulus of this expression is $\left\{\left(1-\frac{\theta' u}{2}\right)^2+\frac{1}{4}\theta'^2u^2\cot^2\theta\right\}^{\frac{1}{2}}$, which is always greater than $\left(1-\frac{\theta' u}{2}\right)$, and therefore always greater than $\frac{1}{2}$; it follows that the modulus of $\left(1+\frac{\theta' u}{z^2-1}\right)^{-m-\frac{1}{2}}$ is always less than $2^{m+\frac{1}{2}}$, hence also the modulus of $\lambda\left(1+\frac{\theta' u}{z^2-1}\right)^{-m-\frac{1}{2}}$ is always less than $2^{m+\frac{1}{2}}$; put $\lambda\left(1+\frac{\theta' u}{z^2-1}\right)^{-m-\frac{1}{2}}=\rho(\cos\chi+\iota\sin\chi)$, where ρ, χ are functions of u, and $\rho<2^{m+\frac{1}{2}}$ for all values of u; we have then

$$\int_0^1 u^{-m-\frac{1}{2}+r}(1-u)^{n+m}\lambda\left(1+\frac{\theta' u}{z^2-1}\right)^{-m-\frac{1}{2}}du=\int_0^1 u^{-m-\frac{1}{2}+r}(1-u)^{n+m}\rho(\cos\chi+\iota\sin\chi)\,du,$$

in this integral the real part and the coefficient of ι in the imaginary part are each less than $2^{m+\frac{1}{2}}\int_0^1 u^{-m-\frac{1}{2}+r}(1-u)^{n+m}du$, hence the modulus of the expression is less than $2^{m+1}\int_0^1 u^{-m-\frac{1}{2}+r}(1-u)^{n+m}du$. Now the $r+1^{\text{th}}$ term of the series (57) is obtained by putting $\theta'=0$, $\lambda=1$ in the expression for the remainder after r terms, it has thus been shown that the modulus of the remainder after r terms is less than 2^{m+1} times the modulus of the $r+1^{\text{th}}$ term, and this is true for all values of θ, not merely for those for which the series is convergent. The two quantities $e^{-m\pi\iota}\,Q_n^m(\cos\theta+0\,.\,\iota)$, $e^{-m\pi\iota}\,Q_n^m(\cos\theta-0\,.\,\iota)$ are conjugate complex quantities, hence the remainders after r terms in the series for $Q_n^m(\cos\theta+0\,.\,\iota)$, $Q_n^m(\cos\theta-0\,.\,\iota)$ are of the form

$$(X\pm\iota Y)\,e^{m\pi\iota}\frac{\Pi\left(-\frac{1}{2}\right)\Pi(n+m)}{\Pi\left(n+\frac{1}{2}\right)}\frac{e^{\mp(n+\frac{1}{2})\iota\theta\mp\iota\pi/4}}{(2\sin\theta)^{\frac{1}{2}}}(-1)^r\frac{1^2-4m^2\ldots\overline{2r-1}|^2-4m^2}{2\,.\,4\ldots 2r\,.\,2n+3\ldots 2n+2r+1}\frac{e^{\mp r\iota(\theta+\pi/2)}}{(2\sin\theta)^r},$$

when X and Y are each less than $2^{m+\frac{1}{2}}$; using (29) we now see that the remainder in the series (56) for $Q_n^m(\cos\theta)$, is of the form

$$(X^2+Y^2)^{\frac{1}{2}}\,.\,\sqrt{\pi}\,.\,\frac{\Pi(n+m)}{\Pi\left(n+\frac{1}{2}\right)}(-1)^r\frac{1^2-4m^2\,.\,3^2-4m^2\ldots(2r-1)^2-4m^2}{2\,.\,4\ldots 2r\,.\,(2n+3)\ldots(2n+2r+1)}$$

$$\frac{\cos\left(\overline{n+\frac{2r+1}{2}}\,\theta+\frac{2r+1}{4}\pi+\frac{m\pi}{2}-\beta\right)}{(2\sin\theta)^{r+\frac{1}{2}}},$$

when β denotes $\tan^{-1}\dfrac{\text{Y}}{\text{X}}$; finally this remainder is numerically less than

$$2^{m+1}\sqrt{\pi}\,.\,\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})}\;\frac{1^2-4m^2\,.\,3^2-4m^2\ldots(2r+1)^2-4m^2}{2\,.\,4\ldots 2r\,.\,(2n+3)\ldots(2n+2r+1)}\;\frac{1}{(2\sin\theta)^{r+\frac{1}{2}}},$$

it has thus been shown that for all real values of m and n such that $n+m-1$, $m+\frac{1}{2}$ are positive, the series (60) may be used to obtain an approximate value of $\text{Q}_n^m(\cos\theta)$ for all values of θ between 0 and π; if the first r terms of the series are taken, the error is certainly less than 2^{m+1} times what we get by writing unity for the cosine in the $r+1$th term, r being any number greater than $m+\frac{1}{2}$. A particular case of this theorem, namely, that in which $m=0$, and n is an integer, has already been obtained otherwise by STIELTJES.*

It has been shown that $\iota\pi e^{m\pi\iota}\,\text{P}_n^m(\mu) = e^{\frac{1}{2}m\pi\iota}\,\text{Q}_n^m(\mu - 0\,.\,\iota) - e^{-\frac{1}{2}m\pi\iota}\,\text{Q}_n^m(\mu + 0\,.\,\iota)$, it therefore follows that the series (59) for $\text{P}_n^m(\cos\theta)$, may, under the same conditions as regards n, m, be used to obtain approximate values of $\text{P}_n^m(\cos\theta)$, the error being limited in the same manner as in the case of (60).

Approximate Values of $\text{P}_n^m(\mu)$, $\text{Q}_n^m(\mu)$ *when* n *is a large real quantity and* μ *is real.*

31. It is well known that when n is a large integer, $\dfrac{\Pi(n)}{\Pi(n+\frac{1}{2})}$ is approximately equal to $\dfrac{1}{\sqrt{n}}$, it follows from (59), (60) that the asymptotic values of $\dfrac{\Pi(n)}{\Pi(n+m)}\,\text{P}_n^m(\cos\theta)$, $\dfrac{\Pi(n)}{\Pi(n+m)}\,\text{Q}_n^m(\cos\theta)$ for a large real integral value of n are given by

$$\frac{\Pi(n)}{\Pi(n+m)}\,\text{P}_n^m(\cos\theta) = \sqrt{\frac{2}{n\pi\sin\theta}}\,\sin\left(\overline{n+\tfrac{1}{2}}\,\theta+\frac{\pi}{4}+\frac{m\pi}{2}\right)$$

$$\frac{\Pi(n)}{\Pi(n+m)}\,\text{Q}_n^m(\cos\theta) = e^{m\pi\iota}\sqrt{\frac{\pi}{2n\sin\theta}}\,\cos\left(\overline{n+\tfrac{1}{2}}\,\theta+\frac{\pi}{4}+\frac{m\pi}{2}\right). \quad . \quad (62).$$

These expressions are generalizations of the known asymptotic values

$$\text{P}_n(\cos\theta) = \sqrt{\frac{2}{n\pi\sin\theta}}\,\sin\left(\overline{n+\tfrac{1}{2}}\,\theta+\frac{\pi}{4}\right)$$

which was given by LAPLACE, and

$$\text{Q}_n(\cos\theta) = \sqrt{\frac{\pi}{2n\sin\theta}}\,\cos\left(\overline{n+\tfrac{1}{2}}\,\theta+\frac{\pi}{4}\right)$$

given by HEINE.†

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[*] 'Annales de la Faculté des Sciences de Toulouse,' vol. 4, in a paper entitled "Sur les Polynômes de Legendre."

[†] 'Kugelfunctionen,' vol. 1, p. 175.

To obtain a closer approximation for large values of n, we use the theorem

$$\Pi(n) = \sqrt{2\pi n}\,.\,e^{-n}n^{n}\left(1+\frac{1}{12n}+\dots\right)$$

we have

$$\frac{\Pi(n)}{\Pi(n+\frac{1}{2})} = \frac{\sqrt{n}\,.\,e^{-n}\,.\,n^{n}\left(1+\frac{1}{12n}\right)}{\sqrt{n+\frac{1}{2}}\,.\,e^{-(n+\frac{1}{2})}\,(n+\frac{1}{2})^{n+\frac{1}{2}}\left(1+\frac{1}{12n+6}\right)}, \text{ approximately}$$

$$= \frac{1}{\sqrt{n}}\left(1+\frac{1}{2n}\right)^{-1} e^{\frac{1}{2}}\,.\left(1+\frac{1}{2n}\right)^{-n}; \text{ neglecting terms in } \frac{1}{n^{\frac{5}{2}}},$$

now

$$\log\left(1+\frac{1}{2n}\right)^{-n} = -n\left(\frac{1}{2n}-\frac{1}{8n^{2}}\right) = -\tfrac{1}{2}+\frac{1}{8n},$$

hence

$$\left(1+\frac{1}{2n}\right)^{-n} = e^{-\frac{1}{2}}\left(1+\frac{1}{8n}\right), \text{ approximately,}$$

or

$$\frac{\Pi(n)}{\Pi(n+\frac{1}{2})} = \frac{1}{\sqrt{n}}\left(1-\frac{1}{2n}\right)\left(1+\frac{1}{8n}\right) = \frac{1}{\sqrt{n}}\left(1-\frac{3}{8n}\right)$$

when terms in $\frac{1}{n^{\frac{5}{2}}}$ . . . are neglected. We thus find as an approximation to $\frac{\Pi(n)}{\Pi(n+m)}\,\mathrm{P}_n^m(\cos\theta)$, by taking the first two terms in (59),

$$\sqrt{\frac{2}{n\pi\sin\theta}}\left(1-\frac{3}{8n}\right)\left\{\sin\left(\overline{n+\tfrac{1}{2}}\,\theta+\frac{\pi}{4}+\frac{m\pi}{2}\right)-\frac{1^{2}-4m^{2}}{4n}\,\frac{1}{2\sin\theta}\cos\left(\overline{n+\tfrac{3}{2}}\,\theta+\frac{\pi}{4}+\frac{m\pi}{2}\right)\right\}$$

or

$$\frac{\Pi(n)}{\Pi(n+m)}\,\mathrm{P}_n^m(\cos\theta) = \sqrt{\frac{2}{n\pi\sin\theta}}\left[\left(1-\frac{1-2m^{2}}{4n}\right)\sin\left(\overline{n+\tfrac{1}{2}}\,\theta+\frac{\pi}{4}+\frac{m\pi}{2}\right)\right.$$
$$\left.-\frac{1-4m^{2}}{8n}\cot\theta\cos\left(\overline{n+\tfrac{1}{2}}\,\theta+\frac{\pi}{4}+\frac{m\pi}{2}\right)\right] \quad (63).$$

Similarly we find

$$\frac{\Pi(n)}{\Pi(n+m)}\,\mathrm{Q}_n^m(\cos\theta) = \sqrt{\frac{\pi}{2n\sin\theta}}\left\{\left(1-\frac{1+2m^{2}}{4n}\right)\cos\left(\overline{n+\tfrac{1}{2}}\,\theta+\frac{\pi}{4}+\frac{m\pi}{2}\right)\right.$$
$$\left.+\frac{1-4m^{2}}{8n}\cot\theta\sin\left(\overline{n+\tfrac{1}{2}}\,\theta+\frac{m\pi}{2}\right)\right\} \quad . \quad (64).$$

In (63), (64), n is large but not necessarily integral, and m is not necessarily integral.

32. When μ is real and greater than unity, let it be denoted by $\cosh\psi$; in Art. 28, $\mathrm{P}_n^m(\mu)$ has been expressed in terms of two hypergeometric series, in both of

which the fourth element is $\frac{1}{1-z^2}$; when $z = e^{\psi}$, this expression for $P_n^m(\mu)$ becomes approximately, when n is large,

$$P_n^m(\cosh\psi) = \frac{1}{\sqrt{\pi}}\left\{\frac{\Pi(n+m)}{\Pi(n)}\cdot\frac{\Pi(n)}{\Pi(n+\frac{1}{2})}\,\frac{\sin(n+m)\pi}{\cos n\pi}\,\frac{e^{-n\psi}}{\sqrt{2e^{\psi}\sinh\psi}}\left(1-\frac{1-4m^2}{4n}\,\frac{e^{-\psi}}{2\sinh\psi}\right)\right.$$
$$\left.+\frac{\Pi(n)}{\Pi(n-m)}\cdot\frac{\Pi(n-\frac{1}{2})}{\Pi(n)}\,\frac{e^{(n+1)\psi}}{\sqrt{2e^{\psi}\sinh\psi}}\left(1+\frac{1-4m^2}{4n}\cdot\frac{e^{-\psi}}{2\sinh\psi}\right)\right\},$$

except when n (supposed positive) is half an odd integer, the first term is very much less than the second, on account of the factor $e^{-n\psi}$; hence

$$\frac{\Pi(n-m)}{\Pi(n)}P_n^m(\cosh\psi) = \frac{1}{\sqrt{\pi n}}\left(1-\frac{1}{8n}\right)\frac{e^{n\psi}}{\sqrt{1-e^{-2\psi}}}\left(1+\frac{1-4m^2}{4n}\cdot\frac{e^{-2\psi}}{1-e^{-2\psi}}\right),$$

or,

$$\frac{\Pi(n-m)}{\Pi(n)}P_n^m(\cosh\psi) = \frac{1}{\sqrt{\pi n}}\cdot\frac{e^{n\psi}}{\sqrt{1-e^{-2\psi}}}\left\{1-\frac{3}{8n}+\frac{m^2}{n}+\frac{1-4m^2}{4n}\cdot\frac{1}{1-e^{-2\psi}}\right\} \quad (65).$$

The asymptotic value of $\frac{\Pi(n-m)}{\Pi(n)}P_n^m(\cosh\psi)$ is therefore $\frac{1}{\sqrt{\pi n}}\cdot\frac{e^{n\psi}}{\sqrt{1-e^{-2\psi}}}$, except in the case in which n is equal to half an odd integer.

From (61) we see that the approximate value of $\frac{\Pi(n)}{\Pi(n+m)}Q_n^m(\cosh\psi)$, for large values of n is

$$e^{m\pi\iota}\sqrt{\pi}\cdot\frac{1}{\sqrt{n}}\left(1-\frac{3}{8n}\right)\frac{e^{-(n+1)\psi}}{\sqrt{1-e^{-2\psi}}}\left\{1-\frac{1^2-4m^2}{4n}\,\frac{e^{-2\psi}}{1-e^{-2\psi}}\right\},$$

or

$$\frac{\Pi(n)}{\Pi(n+m)}Q_n^m(\cosh\psi) = e^{m\pi\iota}\sqrt{\frac{\pi}{n}}\cdot\frac{e^{-(n+1)\psi}}{\sqrt{1-e^{-2\psi}}}\left\{1-\frac{1}{8n}+\frac{m^2}{n}-\frac{1-4m^2}{4n}\cdot\frac{1}{1-e^{-2\psi}}\right\} \quad (66),$$

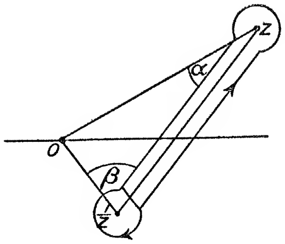
the asymptotic value is $e^{m\pi\iota}\sqrt{\frac{\pi}{n}}\cdot\frac{e^{-(n+1)\psi}}{\sqrt{1-e^{-2\psi}}}$.

It may be remarked that the semi-convergent expressions for BESSEL'S functions $J_m(x)$, $Y_m(x)$ may be obtained from the series (59), (60), by putting $\theta = x/n$ and proceeding the limit $n = \infty$.

Expressions for $P_n^m(\mu)$, as definite integrals taken along real paths.

33. In (50) change m into $-m$, we have then

$$P_n^{-m}(\mu) = \frac{1}{2\pi\iota}\cdot\frac{1}{2^m}\cdot\frac{\pi\sec m\pi}{\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}(\mu^2-1)^{-\frac{1}{2}m}\int^{(z+,\,1/z-)}h^{n-m}(1-2\mu h+h^2)^{m-\frac{1}{2}}\,dh\,;$$



now suppose the real part of $m+\frac{1}{2}$ is positive, then provided the real part of μ is positive the figure is as in Art. 24 ; we may take the path of integration to be two straight lines on opposite sides of the straight line joining the points $z, \frac{1}{z}$, and two indefinitely small circles round these points ; the integrals round these circles will vanish. If the real part of μ had been negative, so that the line joining $z, \frac{1}{z}$, were on the left of 0, the path of the integral (50) must have been placed so that 0 was on its left hand, and thus we could not have reduced the integral to integrals along the line joining $z, \frac{1}{z}$; it is therefore essential in what follows that the real part of μ be supposed to be positive.

We have now

$$\int^{(z+,\,1/z-)} h^{n-m}(1-2\mu h+h^2)^{m-\frac{1}{2}}\,dh = (e^{-\iota\pi\,\overline{2m-1}}-1)\int_{1/z}^{z} h^{n-m}(1-2\mu h+h^2)^{m-\frac{1}{2}}\,dh,$$

where the phases of $1-hz$, $1-\frac{h}{z}$ in the integral on the right-hand side, are $2\pi-\beta$, and α respectively, we thus have

$$P_n^{-m}(\mu)=\frac{1}{2^m}\cdot\frac{e^{-\iota\pi(m-\frac{1}{2})}}{\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}(\mu^2-1)^{-\frac{1}{2}m}\int_{1/z}^{z} h^{n-m}(1-2\mu h+h^2)^{m-\frac{1}{2}}\,dh.$$

Let $h=\mu+(\mu^2-1)^{\frac{1}{2}}\cos\psi$, then

$$dh=-\sqrt{\mu^2-1}\,\sin\psi\,d\psi$$

$$1-2\mu h+h^2=-(\mu^2-1)\sin^2\psi=e^{+\iota\pi}(\mu^2-1)\sin^2\psi$$

since the phase of $1-2\mu h+h^2$ is $\alpha-\beta+2\pi$, which is $2\lambda+\pi$, and the phase of μ^2-1 is 2λ; thus

$$P_n^{-m}(\mu)=\frac{(\mu^2-1)^{\frac{1}{2}m}}{2^m\,\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\int_0^{\pi}(\mu+\sqrt{\mu^2-1}\cos\psi)^{n-m}\sin^{2m}\psi\,d\psi \quad . \quad (67).$$

Or, using equation (19), we have

$$\mathrm{P}_n^m(\mu) - \frac{2}{\pi} e^{-m\pi\iota} \sin m\pi \,.\, \mathrm{Q}_n^m(\mu)$$
$$= \frac{\Pi(n+m)}{\Pi(n-m)} \frac{(\mu^2-1)^{\frac{1}{2}m}}{2^m \Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} \int_0^\pi (\mu + \sqrt{\mu^2-1} \cos\psi)^{n-m} \sin^{2m}\psi \, d\psi \quad (68).$$

This relation holds for all values of n and m, subject to the conditions that the real parts of $m+\frac{1}{2}$, μ are positive; the phase of $\mu + \sqrt{\mu^2-1} \cos\psi$ is the same as that of μ when $\psi = \frac{1}{2}\pi$.

In (68) change n into $-n-1$, we then have, on using the relation (18), after some reduction

$$\mathrm{P}_n^m(\mu) - \frac{2}{\pi} e^{-m\pi\iota} \sin m\pi \,.\, \mathrm{Q}_n^m(\mu)$$
$$= \frac{\Pi(n+m)}{\Pi(n-m)} \frac{(\mu^2-1)^{\frac{1}{2}m}}{2^m \Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} \int_0^\pi \frac{\sin^{2m}\psi}{(\mu + \sqrt{\mu^2-1}\cos\psi)^{n+m+1}} d\psi \quad . \quad (69).$$

34. From (68), (69) it is easy to find the corresponding formulæ for the case in which the real part of μ is negative; in this case we have

$$\mathrm{P}_n^m(-\mu) - \frac{2}{\pi} e^{-m\pi\iota} \sin m\pi \,.\, \mathrm{Q}_n^m(-\mu)$$
$$= \frac{\Pi(n+m)}{\Pi(n-m)} \frac{e^{\mp m\pi\iota}}{2^m \Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} (\mu^2-1)^{\frac{1}{2}m} \int_0^\pi e^{\mp(n-m)\pi\iota} (\mu + \sqrt{\mu^2-1}\cos\psi)^{n-m} \sin^{2m}\psi \, d\psi.$$

The expression on the left-hand side is equal to

$$e^{\mp n\pi\iota} \mathrm{P}_n^m(\mu) - \frac{2\sin(n+m)\pi}{\pi} e^{-m\pi\iota} \mathrm{Q}_n^m(\mu) + \frac{2}{\pi} e^{-m\pi\iota} \sin m\pi \,.\, e^{\pm n\pi\iota} \mathrm{Q}_n^m(\mu);$$

hence

$$\mathrm{P}_n^m(\mu) - \frac{2}{\pi} e^{-m\pi\iota} \sin n\pi \,.\, e^{\pm(n-m)\pi\iota} \mathrm{Q}_n^m(\mu)$$
$$= \frac{\Pi(n+m)}{\Pi(n-m)} \frac{1}{2^m \Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} (\mu^2-1)^{\frac{1}{2}m} \int_0^\pi (\mu + \sqrt{\mu^2-1}\cos\psi)^{n-m} \sin^{2m}\psi \, d\psi \quad (70),$$

where the upper or lower sign is to be taken according as the imaginary part of μ is positive or negative; (70) corresponds to (68).

In a similar manner we find, corresponding to (69),

$$-e^{\mp 2n\pi\iota} \mathrm{P}_n^m(\mu) + \frac{2}{\pi} e^{-m\pi\iota} \sin m\pi \,.\, e^{\mp(n+m)\pi\iota} \mathrm{Q}_n^m(\mu)$$
$$= \frac{\Pi(n+m)}{\Pi(n-m)} \frac{1}{2^m \Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} (\mu^2-1)^{\frac{1}{2}m} \int_0^\pi \frac{\sin^{2m}\psi}{(\mu + \sqrt{\mu^2-1}\cos\psi)^{n+m+1}} d\psi \quad (71),$$

where, as before, the upper or lower sign is taken according as the imaginary part of μ is positive or negative.

35. When $\mu = \cos\theta$, and θ lies between 0 and $\frac{1}{2}\pi$, the expression on the left-hand side of (68) becomes, on putting $\mu = \cos\theta + 0 \,.\, \iota$,

$$e^{-\frac{1}{2}m\pi\iota}\mathrm{P}_n^m(\cos\theta) - \frac{2}{\pi}e^{-m\pi\iota}\sin m\pi \,.\, e^{\frac{3}{2}m\pi\iota}\left\{\mathrm{Q}_n^m(\cos\theta) - \frac{\iota\pi}{2} \,.\, \mathrm{P}_n^m(\cos\theta)\right\},$$

and on the right-hand side $(\mu^2 - 1)^{\frac{1}{2}m} = e^{\frac{1}{2}m\pi\iota}\sin^m\theta$, hence (68) becomes

$$\begin{aligned}\cos m\pi \,.\, \mathrm{P}_n^m(\cos\theta) &- \frac{2}{\pi}\sin m\pi \,.\, \mathrm{Q}_n^m(\cos\theta) \\ &= \frac{\Pi(n+m)}{\Pi(n-m)}\frac{1}{2^m\Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})}\sin^m\theta\int_0^\pi(\cos\theta + \iota\sin\theta\cos\psi)^{n-m}\sin^{2m}\psi\,d\psi \,. \quad (72).\end{aligned}$$

Again, on putting $\mu = \cos\theta - 0 \,.\, \iota$, we find in a similar manner

$$\begin{aligned}\cos m\pi \,.\, \mathrm{P}_n^m(\cos\theta) &- \frac{2}{\pi}\sin m\pi \,.\, \mathrm{Q}_n^m(\cos\theta) \\ &= \frac{\Pi(n+m)}{\Pi(n-m)}\frac{1}{2^m\Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})}\sin^m\theta\int_0^\pi(\cos\theta - \iota\sin\theta\cos\psi)^{n-m}\sin^{2m}\psi\,d\psi \,. \quad (73).\end{aligned}$$

Again, putting $\mu = \cos\theta + 0 \,.\, \iota$ in (55), we have

$$\begin{aligned}\cos m\pi \,.\, \mathrm{P}_n^m(\cos\theta) &- \frac{2}{\pi}\sin m\pi \,.\, \mathrm{Q}_n^m(\cos\theta) \\ &= \frac{\Pi(n+m)}{\Pi(n-m)}\frac{\sin^m\theta}{2^m\Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})}\int_0^\pi\frac{\sin^{2m}\psi}{(\mu \pm \sqrt{\mu^2-1}\cos\psi)^{n+m+1}}\,d\psi \quad . \quad (74).\end{aligned}$$

Next let us consider the case in which θ lies between $\frac{1}{2}\pi$ and π; we find from (70), by putting $\mu = \cos\theta \pm 0 \,.\, \iota$,

$$\begin{aligned}e^{\mp m\pi\iota} \,.\, \mathrm{P}_n^m(\cos\theta)\{1 \pm \iota\sin n\pi \,.\, e^{\pm n\pi\iota}\} &- \frac{2}{\pi}\sin n\pi \,.\, e^{\pm(n-m)\pi\iota}\,\mathrm{Q}_n^m(\cos\theta) \\ &= \frac{\Pi(n+m)}{\Pi(n-m)}\frac{\sin^m\theta}{2^m\Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})}\int_0^\pi(\cos\theta \pm \iota\sin\theta\cos\psi)^{n-m}\sin^{2m}\psi\,d\psi \quad . \quad (75),\end{aligned}$$

this corresponds to (72), (73); the phase of $\cos\theta \pm \iota\sin\theta\cos\psi$, when $\psi = \frac{1}{2}\pi$, is $+\pi$ or $-\pi$ according as the upper or the lower signs are taken in the exponentials.

Again, corresponding to (74), we find that when θ lies between $\frac{1}{2}\pi$ and π,

$$\begin{aligned}-e^{\mp(n+m)\pi\iota} \,.\, \mathrm{P}_n^m(\cos\theta)\{e^{\mp n\pi\iota} + \iota\sin n\pi\} &+ \frac{2}{\pi}\sin n\pi \,.\, e^{\mp(n+m)\pi\iota}\,\mathrm{Q}_n^m(\cos\theta) \\ &= \frac{\Pi(n+m)}{\Pi(n-m)}\frac{\sin^m\theta}{2^m\Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})}\int_0^\pi\frac{\sin^{2m}\psi}{(\cos\theta \pm \iota\sin\theta\cos\psi)^{n+m+1}}\,d\psi \quad . \quad (76),\end{aligned}$$

where, as before, the phase of $\cos\theta \pm \iota \sin\theta \cos\psi$ is $\pm \pi$, when $\psi = \frac{1}{2}\pi$, according as the upper or lower signs are taken in the exponentials.

36. In the important case in which m is a positive integer, we find, from (68) and (70), that

$$\frac{\Pi(n+m)}{\Pi(n-m)} \frac{(\mu^2-1)^{\frac{1}{2}m}}{2^m \Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} \int_0^\pi (\mu + \sqrt{\mu^2-1}\cos\psi)^{n-m} \sin^{2m}\psi \, d\psi$$

is equal to

$$\mathrm{P}_n^m(\mu), \quad \text{or} \quad \mathrm{P}_n^m(\mu) - \frac{2}{\pi} e^{\pm n\pi\iota} \sin n\pi \,.\, \mathrm{Q}_n^m(\mu) \quad . \quad . \quad . \quad . \quad (77),$$

according as the real part of μ is positive or negative.

From (69), (71), we find in this case

$$\mathrm{P}_n^m(\mu) = \frac{\Pi(n+m)}{\Pi(n-m)} \frac{(\mu^2-1)^{\frac{1}{2}m}}{2^m \Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} \int_0^\pi \frac{\sin^{2m}\psi}{(\mu + \sqrt{\mu^2-1}\cos\psi)^{n+m+1}} d\psi,$$

when the real part of μ is positive, and

$$-e^{\mp 2n\pi\iota} \mathrm{P}_n^m(\mu) + \frac{2}{\pi} \sin n\pi \,.\, \mathrm{Q}_n^m(\mu)$$
$$= \frac{\Pi(n+m)}{\Pi(n-m)} \frac{(\mu^2-1)^{\frac{1}{2}m}}{2^m \Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} \int_0^\pi \frac{\sin^{2m}\psi}{(\mu + \sqrt{\mu^2-1}\cos\psi)^{n+m+1}} d\psi \quad . \quad . \quad (78),$$

when the real part of μ is negative, the upper or lower sign in the exponential being taken according as the imaginary part of μ is positive or negative.

When $\mu = \cos\theta$, we have in the case in which m is a positive integer,

$$\mathrm{P}_n^m(\cos\theta) = (-1)^m \frac{\Pi(n+m)}{\Pi(n-m)} \frac{\sin^m\theta}{2^m \Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} \int_0^\pi (\cos\theta \pm \iota \sin\theta \cos\psi)^{n-m} \sin^{2m}\psi \, d\psi. \quad (79),$$

when θ may have any value between 0 and π.

Also

$$\mathrm{P}_n^m(\cos\theta) = (-1)^m \frac{\Pi(n+m)}{\Pi(n-m)} \frac{\sin^m\theta}{2^m \Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} \int_0^\pi \frac{\sin^{2m}\psi}{(\cos\theta \pm \iota\sin\theta\cos\psi)^{n+m+1}} d\psi,$$

where θ lies between 0 and $\frac{1}{2}\pi$, and

$$-e^{\mp n\pi\iota} \mathrm{P}_n^m(\cos\theta)\,(e^{\mp n\pi\iota} + \iota \sin n\pi) + \frac{2}{\pi} e^{\mp n\pi\iota} \sin n\pi \,.\, \mathrm{Q}_n^m(\cos\theta)$$
$$= (-1)^m \frac{\Pi(n+m)}{\Pi(n-m)} \frac{\sin^m\theta}{2^m \Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} \int_0^\pi \frac{\sin^{2m}\psi}{(\cos\theta \pm \iota\sin\theta\cos\psi)^{n+m+1}} d\psi \quad . \quad (80),$$

where θ lies between $\frac{1}{2}\pi$ and π.

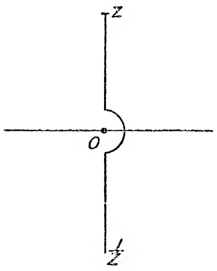
Remarks on HEINE'S *definition of* $P_n(\mu)$.

37. HEINE has proposed* to define the function $P_n(\mu)$ for complex values of n and μ, by means of the expression

$$P_n(\mu) = \frac{1}{\pi}\int_0^\pi (\mu + \sqrt{\mu^2 - 1}\cos\psi)^n\,d\psi.$$

It will appear from what we have shown in Arts. 33–36, that this definition is not a valid one, as the function given by the definite integral for values of μ with a negative real part is not the analytical continuation of the function given by the same definite integral for values of μ with a positive real part; it follows that $P_n(\mu)$ can be defined by the above expression only for values of μ with a positive real part.

The fact that the definite integral is of ambiguous meaning at the imaginary μ axis is clear if we attend to the phases of the integrand $(\mu + \sqrt{\mu^2 - 1}\cos\psi)^n$, or h^n; μ being purely imaginary there is a value of ψ between 0 and π for which h vanishes, and in passing through this value of ψ the phase of the integrand changes by a finite amount. The h integral in Art. 33 is taken along a path joining $z, \frac{1}{z}$ which has the point $h = 0$ on the left hand side, thus for purely imaginary values of μ the path may be placed



as in the figure, with a semi-circular portion to avoid the point $h = 0$; we thus see that in the above definite integral there must be a sudden diminution of phase $n\pi$ in the integrand as $\cos\psi$ passes through the value $\frac{-\mu}{\sqrt{\mu^2 - 1}}$; if this be taken into account the definite integral will represent the function $P_n(\mu)$ for purely imaginary values of μ; there is however nothing in the definite integral itself which decides apart from convention what the change of phase in the integrand shall be as it passes through its zero value.

Next suppose μ to cross the imaginary axis, the h integral can then be taken from $\frac{1}{z}$ to z along a loop round the point $h = 0$, and then along a straight line to the point $h = z$, but cannot be taken directly from $\frac{1}{z}$ to z; it thus appears that the function $\mathrm{P}_n(\mu)$ is no longer represented by the definite integral, but that the value of the definite integrals involves $\mathrm{Q}_n(\mu)$ as well as $\mathrm{P}_n(\mu)$; in fact, we have shown in (70) that in this case

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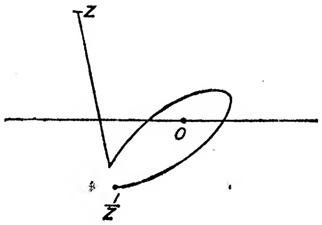
* 'Kugelfunctionen.,' Vol. 1, p. 37.



$$\frac{1}{\pi}\int_0^\pi (\mu + \sqrt{\mu^2 - 1}\cos\psi)^n\, d\psi = \mathrm{P}_n(\mu) - \frac{2}{\pi} e^{\pm n\pi\iota} \sin n\pi \,.\, \mathrm{Q}_n(\mu),$$

where the upper or lower sign is to be taken in the exponential according as the imaginary part of μ is positive or negative.

The only case in which HEINE's definition is valid for all values of μ is when n is a real integer.

HEINE deduces from his definition that for unrestricted values of n, the function $\mathrm{P}_n(\mu)$ is represented when mod $\mu > 1$, by the series

$$\frac{1}{2^n} \cdot \frac{\Pi(2n)}{\Pi(n)\,\Pi(n)}\, \mu^n\, \mathrm{F}\left(-\frac{n}{2}, \frac{1-n}{2}, \tfrac{1}{2} - n, \frac{1}{\mu^2}\right),$$

this result, following from the incorrect definition, is erroneous, the correct expression being given by (23) and involving two hypergeometric series.

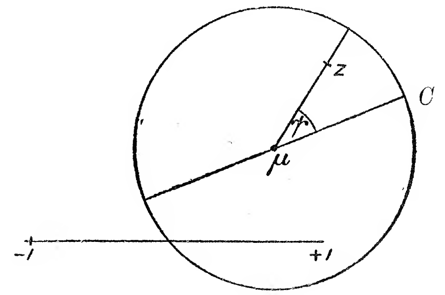
It was to be expected *à priori* that as $\mathrm{P}_n(\mu)$ was defined by means of an integral taken along a path containing the singular points μ and $+1$, but excluding -1, the function so defined would not in general possess any kind of symmetry about the imaginary axis.

Definite Integral Expressions for $\mathrm{P}_n^m(\mu)$ *when* m *is a Real Integer.*

38. When m is a real integer, the formula (4) for $\mathrm{P}_n^m(\mu)$ becomes

$$\mathrm{P}_n^m(\mu) = \frac{1}{2\pi\iota}\, \frac{\Pi(n+m)}{\Pi(n)}\, \frac{1}{2^n}\, (\mu^2 - 1)^{m/2} \int^{(\mu+,\, 1+)} (t^2 - 1)^n\, (t - \mu)^{-n-m-1}\, dt.$$

Suppose the real part of μ to be positive, and the path of integration to be a circle with centre at the point μ, and of radius greater than mod $(\mu - 1)$ and less than



mod $(\mu + 1)$. On this circle take a point C such that the angle between μz and μC is ψ, and let ϕ be the angular distance of any point of the circle from C. If we put $t = \mu + \sqrt{\mu^2 - 1}\, e^{\iota(\phi-\psi)\mp u}$, the point t represents, for different values of ϕ, points on a circle of centre μ and radius $e^{\mp u}$ mod $(\sqrt{\mu^2 - 1})$; we must thus take u to be such that $e^{\mp u}$ mod $(\sqrt{\mu^2 - 1}) >$ mod $(\mu - 1)$, and $<$ mod $(\mu + 1)$, or $u < \frac{1}{2} \log \text{mod} \frac{\mu+1}{\mu-1}$. Take the circle commencing at C to be the path of integration; we have

$$t^2 - 1 = 2\sqrt{\mu^2 - 1} \, . \, e^{\iota(\phi-\psi)\mp u} [\mu + \sqrt{\mu^2 - 1} \cos(\phi - \psi \pm \iota u)].$$

Hence we have

$$\mathrm{P}_n^m(\mu) = \frac{1}{2\pi} \frac{\Pi(n+m)}{\Pi(n)} \int_0^{2\pi} \{\mu + \sqrt{\mu^2 - 1} \cos(\phi - \psi \pm \iota u)\}^n e^{-m\iota(\phi-\psi)\pm mu} \, d\phi,$$

or

$$\frac{1}{2\pi} \int_0^{2\pi} \{\mu + \sqrt{\mu^2 - 1} \cos(\phi - \psi \pm \iota u)\}^n (\cos m\phi - \iota \sin m\phi) \, d\phi$$
$$= \mathrm{P}_n^m(\mu) \, \frac{\Pi(n)}{\Pi(n+m)} e^{-m\iota(\psi \mp \iota u)}.$$

On changing m into $-m$, and remembering that

$$\mathrm{P}_n^{-m}(\mu) = \frac{\Pi(n-m)}{\Pi(n+m)} \mathrm{P}_n^m(\mu)$$

we have

$$\frac{1}{2\pi} \int_0^{2\pi} \{\mu + \sqrt{\mu^2 - 1} \cos(\phi - \psi \pm \iota u)\}^n (\cos m\phi + \iota \sin m\phi) \, d\phi$$
$$= \mathrm{P}_n^m(\mu) \cdot \frac{\Pi(n)}{\Pi(n+m)} e^{+m\iota(\psi \mp \iota u)},$$

we thus obtain the formulæ

$$\frac{1}{2\pi}\int_0^{2\pi}\{\mu+\sqrt{\mu^2-1}\cos(\phi-\psi\pm\iota u)\}^n \genfrac{}{}{0pt}{}{\cos}{\sin} m\phi\,d\phi$$
$$= \mathrm{P}_n^m(\mu)\frac{\Pi(n)}{\Pi(n+m}\genfrac{}{}{0pt}{}{\cos}{\sin} m(\psi\mp\iota u) \quad . \quad . \quad . \quad . \quad . \quad . \quad (81).$$

If we change n into $-(n+1)$, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\genfrac{}{}{0pt}{}{\cos}{\sin} m\phi}{\{\mu+\sqrt{\mu^2-1}\cos(\phi-\psi\pm\iota u)\}^{n+1}}\,d\phi$$
$$= \mathrm{P}_n^m(\mu)\frac{\Pi(n-m)}{\Pi(n)}(-1)^m\genfrac{}{}{0pt}{}{\cos}{\sin} m(\psi\mp\iota u) \quad . \quad . \quad . \quad . \quad (82).$$

In these formulæ n is unrestricted, m is a real integer, and u is any real positive quantity less than $\frac{1}{2}\log\operatorname{mod}\frac{\mu+1}{\mu-1}$, and the real part of μ is positive.

Formulæ corresponding to (81), (82) have been given by HEINE in the case in which n is a positive integer.*

If in (81), (82) we put $u=0$, $\psi=0$, we have

$$\frac{1}{2\pi}\int_0^{2\pi}(\mu+\sqrt{\mu^2-1}\cos\phi)^n\genfrac{}{}{0pt}{}{\cos}{\sin} m\phi\,d\phi = \frac{\Pi(n)}{\Pi(n+m)}\mathrm{P}_n^m(\mu) \quad . \quad . \quad . \quad (83).$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\genfrac{}{}{0pt}{}{\cos}{\sin} m\phi}{(\mu+\sqrt{\mu^2-1}\cos\phi)^{n+1}}\,d\phi = (-1)^m\frac{\Pi(n-m)}{\Pi(n)}\mathrm{P}_n^m(\mu) \quad . \quad . \quad (84).$$

Definite Integral Expressions for $\mathrm{Q}_n^m(\mu)$.

39. When the real parts of $m+\frac{1}{2}$, $n-m+1$ are both positive, the formula (54), for $\mathrm{Q}_n^m(\mu)$ reduces to

$$\mathrm{Q}_n^m(\mu) = e^{m\pi\iota}\,.\,2^m\frac{\Pi(n+m)}{\Pi(n-m)}\frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})}\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}\int_0^1 u^{m-\frac{1}{2}}(1-u)^{n-m}\left(1-\frac{u}{z^2}\right)^{-n-m-1}du$$

on changing the independent variable to v, where $u=\frac{v-1}{v+1}$, we then have

$$\mathrm{Q}_n^m(\mu) = e^{m\pi\iota}\,.\,2^m\,.\,\frac{\Pi(n+m)}{\Pi(n-m)}\,.\,\frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})}\,.\,(\mu^2-1)^{\frac{1}{2}m}\,.\,2^{n-m+1}$$
$$\int_1^\infty(v^2-1)^{m-\frac{1}{2}}\left\{z+\frac{1}{z}+v\left(z-\frac{1}{z}\right)\right\}^{-n-m-1}dv,$$

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* See 'Kugelfunctionen,' vol. 1, p. 211.

or, on putting $v = \cosh w$, this becomes

$$Q_n^m(\mu) = \frac{1}{2^m} e^{m\pi\iota} \frac{\Pi(n+m)}{\Pi(n-m)} \cdot \frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})} (\mu^2-1)^{\frac{1}{2}m}$$

$$\int_0^\infty \{\mu + \sqrt{\mu^2-1}\cosh w\}^{-n-m-1} \sinh^{2m} w \, dw \quad . \quad (85),$$

where the real parts of $m + \frac{1}{2}$, $n - m + 1$ are positive.

If $m = 0$, we have

$$Q_n(\mu) = \int_0^\infty \{\mu + \sqrt{\mu^2-1}\cosh w\}^{-n-1} dw \quad . \quad . \quad . \quad . \quad . \quad (86)$$

The particular case of (85), when m and n are real integers, is given by HEINE.*

When μ has a real value less than unity, we have, on using (31),

$$Q_n^m(\cos\theta) = \frac{1}{2^{m+1}} \frac{\Pi(n+m)}{\Pi(n-m)} \frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})} \sin^m\theta \left\{ \int_0^\infty \frac{\sinh^{2m} w}{(\cos\theta + \iota\sin\theta\cosh w)^{n+m+1}} dw \right.$$

$$\left. + \int_0^\infty \frac{\sinh^{2m} w}{(\cos\theta - \iota\sin\theta\cosh w)^{n+m+1}} dw \right\};$$

and from (30),

$$P_n^m(\cos\theta) = \frac{1}{2^m \cdot \iota\pi} \cdot \frac{\Pi(n+m)}{\Pi(n-m)} \frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})} \sin^m\theta \left\{ \int_0^\infty \frac{\sinh^{2m} w}{(\cos\theta - \iota\sin\theta\cosh w)^{n+m+1}} dw \right.$$

$$\left. - \int_0^\infty \frac{\sinh^{2m} w}{(\cos\theta + \iota\sin\theta\cosh w)^{n+m+1}} dw \right\}.$$

40. In the formula

$$Q_n^m(\mu) = e^{m\iota\pi} . 2^m . \frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})} \cos m\pi \, (\mu^2-1)^{\frac{1}{2}m} \int_0^{\frac{1}{z}} \frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}} dh \quad . \quad (43),$$

which holds, provided the real parts of $n + m + 1$, $\frac{1}{2} - m$ are positive; put $h = \mu - \sqrt{\mu^2-1}\cosh w$, then when $h = 0$, we have $w = w_0 = \frac{1}{2}\log_e \frac{\mu+1}{\mu-1}$, and when $h = \frac{1}{z}$, $w = 0$, thus since $1 - 2h\mu + h^2 = (\mu^2-1)\sinh^2 v$, we have

$$Q_n^m(\mu)$$

$$= e^{m\pi\iota} . 2^m . \frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})} \cos m\pi . (\mu^2-1)^{-\frac{1}{2}m} \int_0^{w_0} (\mu - \sqrt{\mu^2-1}\cosh w)^{n+m} \sinh^{-2m} w \, dw \quad (87),$$

where $w_0 = \frac{1}{2}\log \text{mod.} \frac{\mu+1}{\mu-1}$, and the real parts of $n + m + 1$, $\frac{1}{2} - m$ are positive.

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* See 'Kugelfunctionen,' vol. 1, p. 222.

If $m = 0$, we have

$$Q_n(\mu) = \int_0^{w_0} (\mu - \sqrt{\mu^2 - 1} \cosh w)^n \, dw \quad . \quad . \quad . \quad . \quad . \quad . \quad (88).$$

It is interesting to compare (87) with the formula obtained by changing m into $-m$, in (85),

$$Q_n^m(\mu)$$
$$= e^{m\pi\iota} \, . \, 2^m \, . \, \frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})} \cos m\pi \, . \, (\mu^2 - 1)^{-\frac{1}{2}m} \int_0^\infty \{\mu + \sqrt{\mu^2 - 1} \cosh w\}^{m-n-1} \sinh^{-2m} w \, dw \quad (89),$$

which holds under the same conditions as (87).

In (87) change m into $-m$, we have then

$$Q_n^m(\mu)$$
$$= e^{m\pi\iota} \, . \, 2^m \frac{\Pi(n+m)}{\Pi(n-m)} \cdot \frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})} (\mu^2 - 1)^{\frac{1}{2}m} \int_0^{w_0} (\mu - \sqrt{\mu^2 - 1} \cosh w)^{n-m} \sinh^{2m} w \, dw \quad (90),$$

which holds when the real parts of $n - m + 1$, $\frac{1}{2} + m$ are positive.

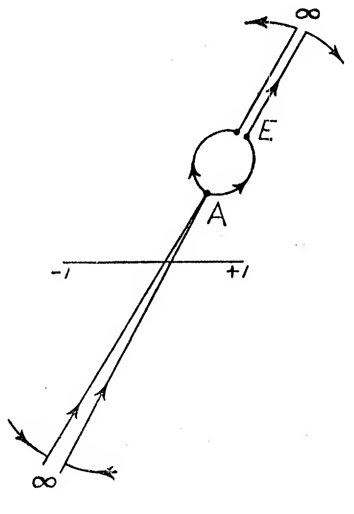
41. In the formula (9), change n into $-n - 1$, we then have

$$Q_{-n-1}^m(\mu) = \frac{e^{n\pi\iota}}{4\iota \sin (n - m)\pi} \cdot \frac{\Pi(n)}{\Pi(n - m)} 2^{n+1} \int^{(-1+,\, 1-)} X \, dt,$$

where

$$X = (\mu^2 - 1)^{\frac{1}{2}m} (t^2 - 1)^{-n-1} (t - \mu)^{n-m}.$$

Place the path as in the figure; starting from A, a semicircle of centre μ is first described, then a straight line from E to ∞, a semicircle of infinite radius, then a straight path from ∞' to A, followed by a similar path taken negatively round the point $+1$.



If the real part of $n - m + 1$ is positive, the integrals along the semicircles with μ as centre vanish when the radius is made indefinitely small. If the real part of $n + m + 1$ is positive, the integrals along the infinite semicircles vanish. We thus have,

$$Q_{-n-1}^{m}(\mu)$$

$$= \frac{e^{n\pi\iota}}{4\iota \sin(n-m)\pi} \frac{\Pi(n)}{\Pi(n-m)} 2^{n+1} \left\{ e^{-n\pi\iota} . 2 \cos m\pi \int_{\mu}^{\infty} X\,dt - e^{-n\pi\iota} . 2 \cos n\pi \int_{\mu}^{\infty'} X\,dt \right\},$$

where in the integrals X commences with the phase it has at A initially. The phase of $t + 1$ at A is $-(2\pi - \gamma)$.

From equation (8), we have

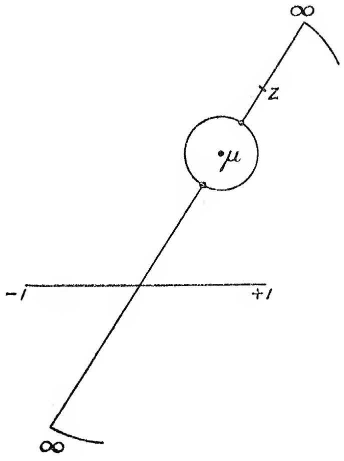
$$P_{n}^{m}(\mu) = \frac{-e^{+n\pi\iota}}{4\pi \sin(n-m)\pi} \cdot \frac{\Pi(n)}{\Pi(n-m)} \cdot 2^{n+1} \int^{(\mu+,\, 1+,\, \mu-,\, 1-)} (t^2 - 1)^{-n-1} (t - \mu)^{n-m}\, dt,$$

where the phase of $t + 1$ in the integrand is γ at A, and thus

$$(t^2 - 1)^{-n-1} (t - \mu)^{n-m} = e^{-2n\pi\iota} X;$$

hence

$$P_{n}^{m}(\mu) = \frac{-e^{-n\pi\iota}}{4\pi \sin(n-m)\pi} \cdot \frac{\Pi(n)}{\Pi(n-m)} 2^{n+1} \int^{(\mu+,\, 1+,\, \mu-,\, 1-)} X\, dt.$$



Taking the path as in the figure, we have, provided the real parts of $n \pm m + 1$ are positive,

$$\int^{(\mu+,\, 1+,\, \mu-,\, 1-)} X\,dt = e^{2(n-m)\pi\iota}\int_{\mu}^{\infty'} X\,dt - e^{(n-3m)\pi\iota}\int_{\mu}^{\infty} X\,dt + e^{-(n+m)\pi\iota}\int_{\mu}^{\infty} X\,dt - \int_{\mu}^{\infty'} X\,dt$$

$$= - e^{-2m\pi\iota} . 2\iota \sin(n-m)\pi \int_{\mu}^{\infty} X\,dt . + e^{(n-m)\pi\iota} . 2\iota \sin(n-m)\pi \int_{\mu}^{\infty'} X\,dt,$$

therefore

$$\mathrm{P}_n^{m}(\mu) = \frac{-e^{-n\pi\iota}}{4\pi} \cdot \frac{\Pi(n)}{\Pi(n-m)} \cdot 2^{n+1}\left\{-2\iota\, e^{-2m\pi\iota}\int_\mu^\infty \mathrm{X}\,dt + 2\iota \,.\, e^{(n-m)\pi\iota}\int_\mu^{\infty'} \mathrm{X}\,dt\right\} \cdot$$

Substituting for $\mathrm{Q}_{-n-1}^{m}(\mu)$ its value in terms of $\mathrm{Q}_n^{m}(\mu)$, $\mathrm{P}_n^{m}(\mu)$ given by (18), we have

$$\begin{aligned}\mathrm{Q}_n^{m}(\mu)\sin(n+m)\pi - \pi\cos n\pi \,.\, e^{m\pi\iota}\,\mathrm{P}_n^{m}(\mu) \\ = \frac{1}{4\iota} \cdot \frac{\Pi(n)}{\Pi(n-m)}\, 2^{n+1}\left\{2\cos m\pi\int_\mu^\infty \mathrm{X}\,dt - 2\cos n\pi\int_\mu^{\infty'} \mathrm{X}\,dt\right\} \cdot\end{aligned}$$

On substituting the value of $\mathrm{P}_n^{m}(\mu)$ in this equation, we have

$$\mathrm{Q}_n^{m}(\mu) = 2^n \,.\, e^{-n\pi\iota}\frac{\Pi(n)}{\Pi(n-m)} \cdot (\mu^2-1)^{\frac{1}{2}m}\int_\mu^\infty (t^2-1)^{-n-1}(t-\mu)^{n-m}\,dt,$$

which holds for all values of n and m, such that the real parts of $n+m+1$, $n-m+1$ are both positive.

In this formula, when t is just greater than μ, the phase of $t-1$ is the same as that of $\mu-1$, but the phase of $t+1$ is less by 2π than that of $\mu+1$, hence if we wish the phase of t^2-1 to be that of μ^2-1, the result must be multiplied by $e^{2n\pi\iota}$; again the phase of $t-\mu$ is that at A, and this is less by π than the phase of $\sqrt{\mu^2-1}$, hence, in order that the phase of $t-\mu$ may be that of $\sqrt{\mu^2-1}$, we must multiply by $e^{-(n-m)\pi\iota}$; the formula now becomes

$$\mathrm{Q}_n^{m}(\mu) = 2^n \,.\, e^{m\pi\iota}\frac{\Pi(n)}{\Pi(n-m)}(\mu^2-1)^{\frac{1}{2}m}\int_\mu^\infty (t^2-1)^{-n-1}(t-\mu)^{n-m}\,dt\,;$$

on substituting $t = \mu + \sqrt{\mu^2-1}\,.\,e^u$, which gives us $\dfrac{t^2-1}{2(t-\mu)} = \mu + \sqrt{\mu^2-1}\cosh u$, where u is a real quantity, we have

$$\mathrm{Q}_n^{m}(\mu) = e^{m\pi\iota} \cdot \frac{\Pi(n)}{\Pi(n-m)} \cdot \tfrac{1}{2}\int_{-\infty}^{\infty} \frac{e^{-mu}}{(\mu+\sqrt{\mu^2-1}\cosh u)^{n+1}}\,du,$$

or

$$\mathrm{Q}_n^{m}(\mu) = e^{m\pi\iota} \cdot \frac{\Pi(n)}{\Pi(n-m)} \cdot \int_0^\infty \frac{\cosh mu}{(\mu+\sqrt{\mu^2-1}\cosh \mu)^{n+1}}\,du \,. \quad . \quad . \quad . \quad (91),$$

where the real parts of $n+m+1$, $n-m+1$ are positive.

In (91), the phase of $\mu+\sqrt{\mu^2-1}\cosh u$ is equal to that of $\mu+\sqrt{\mu^2-1}$ when $u=0$. The formula (91) is a generalization of the formula given by HEINE;*

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* 'Kugelfunctionen,' vol. 1, p. 223.

his formula he proves, for the case in which m is a real integer, by a method of transformation which cannot be applied to obtain the more general result (91).

42. In the formula

$$\mathrm{Q}_n^m(\mu) = \frac{e^{m\pi\iota}}{2^{n+1}} \frac{\Pi(n+m)}{\Pi(n)} (\mu^2-1)^{\frac{1}{2}m} \int_{-1}^{1} (1-t^2)^n (\mu - t)^{-n-m-1}\, dt \quad . \quad . \quad (11),$$

which holds, provided the real part of $n+1$ is positive; let

$$t = \mu - \sqrt{\mu^2-1}\,.\,e^u, \text{ then } 1 - t^2 = \sqrt{\mu^2-1}\,.\,e^u \{2\mu - 2\sqrt{\mu^2-1}\cosh u\},$$

hence

$$\mathrm{Q}_n^m(\mu)$$

$$= \frac{e^{m\pi\iota}}{2^{n+1}} \cdot \frac{\Pi(n+m)}{\Pi(n)} (\mu^2-1)^{\frac{1}{2}m} \int_{\log_e\sqrt{\frac{\mu+1}{\mu-1}}}^{\log_e\sqrt{\frac{\mu-1}{\mu+1}}} \frac{(\mu^2-1)^{\frac{1}{2}n}\,.\,e^{nu}\{2\mu - 2\sqrt{\mu^2-1}\cosh\mu\}^n}{(\mu^2-1)^{\frac{n+m+1}{2}}\, e^{(n+m+1)u}} (-\sqrt{\mu^2-1}\,.\,e^u)\,du,$$

or

$$\mathrm{Q}_n^m(\mu) = e^{m\pi\iota}\,.\,\frac{\Pi(n+m)}{\Pi(n)} \int_0^{\log_e\sqrt{\frac{\mu+1}{\mu-1}}} \{\mu - \sqrt{\mu^2-1}\cosh u\}^n \cosh mu\, du \quad . \quad (92).$$

This formula holds for all values of n and m such that the real part of $n+1$ is positive.

The Evaluation of a certain Definite Integral.

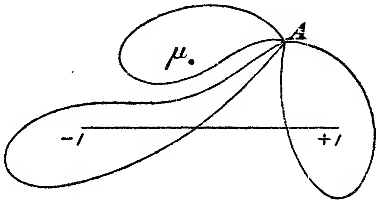
43. Suppose n and m are such that $n-m$ is a real integer, and that they are otherwise unrestricted; in this case the integral

$$\frac{1}{2\pi\iota} (\mu^2-1)^{\frac{1}{2}m} \int \frac{1}{2^n} (t^2-1)^n (t-\mu)^{-n-m-1}\, dt$$

taken round a closed path which includes the three singular points 1, -1, μ will satisfy the fundamental equation (2), since the integrand attains its original value after description of the closed path. We shall take the path to be a circle with centre at the point μ; if we put $t = \mu + \sqrt{\mu^2-1}\,.\,e^{\iota(\phi-\psi)\mp u}$, as in Art. 38, in this case we must have $u > \frac{1}{2}\log \operatorname{mod} \frac{\mu+1}{\mu-1}$, and the integral becomes

$$\frac{1}{2\pi}\int_0^{2\pi} \{\mu + \sqrt{\mu^2-1}\cos(\phi - \psi \pm \iota u)\}^n e^{-m\iota(\phi-\psi)\pm mu}\, d\phi.$$

This integral has been evaluated in Art. 38, when $u < \frac{1}{2}\log\bmod\frac{\mu+1}{\mu-1}$; we proceed to evaluate it in the present case $u > \frac{1}{2}\log\bmod\frac{\mu+1}{\mu-1}$. We shall denote the definite integral by $\frac{1}{2\pi\iota}\,\mathrm{I}(n, m)$.



Denote by L, M, N the integrals of $\frac{1}{2^n}(t^2-1)^n(t-\mu)^{-n-m-1}$ taken along loops from the point A, round the points -1, 1, μ respectively, then

$$(\mu^2-1)^{\frac{1}{2}m}\int^{(\mu+,\,1+,\,\mu-,\,1-)}\frac{1}{2^n}(t^2-1)^n(t-\mu)^{-n-m-1}\,dt = \mathrm{N}+\mathrm{M}e^{-4n\pi\iota}-\mathrm{N}e^{2n\pi\iota}-\mathrm{M}$$
$$= (1-e^{2n\pi\iota})\{\mathrm{N}+\mathrm{M}e^{-4n\pi\iota}(1+e^{2n\pi\iota})\}$$

$$(\mu^2-1)^{\frac{1}{2}m}\int^{(-1+,\,+1-)}\frac{1}{2^n}(t^2-1)^n(t-\mu)^{-n-m-1}\,dt = (\mathrm{L}-\mathrm{M})e^{-2n\pi\iota}.$$

Also

$$\mathrm{I}(n, m) = \mathrm{N}+\mathrm{L}e^{-4\pi n\iota}+\mathrm{M}e^{-2\pi n\iota},$$

hence

$$(1-e^{2\pi n\iota})\,\mathrm{I}(n, m) = (\mu^2-1)^{\frac{1}{2}m}\int^{(\mu+,\,1+,\,\mu-,\,1-)}\frac{1}{2^n}(t^2-1)^n(t-\mu)^{-n-m-1}\,dt$$
$$-(1-e^{-2\pi n\iota})(\mu^2-1)^{\frac{1}{2}m}\int^{(-1+,\,1-)}\frac{1}{2^n}(t^2-1)^n(t-\mu)^{-n-m-1}\,dt,$$

or

$$-e^{n\pi\iota}\,.\,2\iota\sin n\pi\,.\,\mathrm{I}(n, m) = \frac{\Pi(n)}{\Pi(n+m)}\,.\,4\pi\sin n\pi\,.\,e^{n\pi\iota}\,\mathrm{P}_n^m(\mu)$$
$$-\frac{\Pi(n)}{\Pi(n+m)}\,.\,8\sin^2 n\pi\,.\,\mathrm{Q}_n^m(\mu),$$

we thus have

$$\frac{1}{2\pi}\int_0^{2\pi}\{\mu+\sqrt{\mu^2-1}\cos(\phi-\psi\pm\iota u)\}^n e^{-m\iota(\phi-\psi)\pm mu}\,d\phi$$
$$= \frac{\Pi(n)}{\Pi(n+m)}\left\{\mathrm{P}_n^m(\mu)-\frac{2}{\pi}e^{-n\pi\iota}\sin n\pi\,\mathrm{Q}_n^m(\mu)\right\} \quad . \quad (93),$$

where $n - m$ is a real integer, and $u > \frac{1}{2} \log_e \bmod \frac{\mu + 1}{\mu - 1}$. It has been shown in Art. 11 that the expression in (93) is zero when $n - m$ is a negative integer.

When m and n are both integers

$$\frac{1}{2\pi}\int_0^{2\pi} \{\mu + \sqrt{\mu^2 - 1} \cos(\phi - \psi \pm \iota u)\}^n e^{-m\iota(\phi-\psi) \pm mu} \, d\phi = \frac{\Pi(n)}{\Pi(n+m)} \mathrm{P}_n^m(\mu) \,. \quad (94),$$

the right-hand side is zero when n and m are positive, and $n < m$, since in this case $\mathrm{P}_n^m(\mu) = 0$.

Next change m into $-m$, in the formula (93), the expression on the right-hand side then becomes

$$\frac{\Pi(n)}{\Pi(n-m)} \left\{ \mathrm{P}_n^{-m}(\mu) - \frac{2}{\pi} e^{-n\pi\iota} \sin n\pi \,.\, \mathrm{Q}_n^{-m}(\mu) \right\}$$

or

$$\frac{\Pi(n)}{\Pi(n+m)} \left\{ \mathrm{P}_n^m(\mu) - \frac{2}{\pi} e^{-m\pi\iota} \sin m\pi \,.\, \mathrm{Q}_n^m(\mu) \right\} - \frac{\Pi(n)}{\Pi(n+m)} \cdot \frac{2}{\pi} \sin n\pi \,.\, e^{-(n+2m)\pi\iota}, \mathrm{Q}_n^m(\mu),$$

which reduces to $\frac{\Pi(n)}{\Pi(n+m)} \mathrm{P}_n^m(\mu)$, since $n + m$ is a real integer; we thus have the formula

$$\frac{1}{2\pi}\int_0^{2\pi} \{\mu + \sqrt{\mu^2 - 1} \cos(\phi - \psi \pm \iota u)\}^n e^{m\iota(\phi-\psi) \mp mu} \, d\phi = \frac{\Pi(n)}{\Pi(n+m)} \mathrm{P}_n^m(\mu) \quad (95),$$

which holds for all values of m and n such that $m + n$ is a real integer; when m and n are positive integers such that $m > n$, we have $\mathrm{P}_n^m(\mu) = 0$, and the integral in (95) vanishes.

44. In (93) change n into $-n-1$, we have then

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-m\iota(\phi-\psi) \pm mu}}{\{\mu + \sqrt{\mu^2 - 1} \cos(\phi - \psi \pm \iota u)\}^{n+1}} \, d\phi$$

$$= \frac{\Pi(-n-1)}{\Pi(m-n-1)} \left\{ \mathrm{P}_n^m(\mu) + \frac{2}{\pi} e^{n\pi\iota} \sin n\pi \, \mathrm{Q}_{-n-1}^m(\mu) \right\}$$

where $m - n$ is a real integer; now, suppose m and n are both real integers, it is then necessary to evaluate the undetermined form on the right-hand side; to do this, suppose the modulus of μ is greater than unity, and substitute the series in powers of $\frac{1}{\mu}$ for the functions $\mathrm{P}_n^m(\mu)$, $\mathrm{Q}_{-n-1}^m(\mu)$; the expression then becomes

$$\frac{1}{\Pi(m-n-1)}\Big\{\frac{\pi\Pi(-n-1)}{\Pi(m+n-1)\,\Pi(-m-n)}\,\frac{1}{2^{n+1}\cos n\pi}\,\frac{1}{\Pi(n+\frac{1}{2})\,\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\mu^{-n-m-1}$$

$$\mathrm{F}\left(\frac{n+m+2}{2},\frac{n+m+1}{2},n+\tfrac{3}{2},\frac{1}{\mu^2}\right)$$

$$+\,2^n\frac{\Pi(-n-1)}{\Pi(n-m)}\,\frac{\Pi(n-\frac{1}{2})}{\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\mu^{n-m}\,\mathrm{F}\left(\frac{m-n+1}{2},\frac{m-n}{2},\tfrac{1}{2}-n,\frac{1}{\mu^2}\right)\Big\}$$

$$-\,\frac{2}{\pi}\,e^{(n+m)\pi\iota}\,2^n\frac{\pi\,\Pi(m-n-1)\,\Pi(-\frac{1}{2})}{\Pi(n)\,\Pi(-n-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\mu^{n-m}\,\mathrm{F}\left(\frac{m-n+1}{2},\frac{m-n}{2},\tfrac{1}{2}-n,\frac{1}{\mu^2}\right)\Big\}.$$

The ratio of the coefficients of the last two terms can easily be shown to be -1, thus the result reduces to

$$\frac{1}{\Pi(m-n-1)}\,\frac{\pi}{\Pi(m+n-1)}\,\frac{(-n-1)\ldots(-n-m+1)}{2^{n+1}\cos n\pi}\,\frac{1}{\Pi(-\frac{1}{2})\,\Pi(n+\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\mu^{-n-m-1}$$

$$\mathrm{F}\left(\frac{n+m+2}{2},\frac{n+m+1}{2},n+\tfrac{3}{2},\frac{1}{\mu^2}\right),$$

or to

$$\frac{(-1)^{n+1}}{\Pi(m-n-1)}\,\frac{1}{\Pi(n+m)\,\Pi(n)}\,\mathrm{Q}_n^m(\mu).$$

This result must hold whether mod μ is greater or less than unity; hence when m and n are real integers

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-m\iota(\phi-\psi)\pm mu}}{\{\mu+\sqrt{\mu^2-1}\cos(\phi-\psi\pm\iota u)\}^{n+1}}\,d\phi=\frac{(-1)^{n+1}}{\Pi(m-n-1)}\,\frac{1}{\Pi(n+m)\,\Pi(n)}\,\mathrm{Q}_n^m(\mu)\ (96),$$

when $m > n$, and is equal to zero when $m \overline{\overline{<}} n$.

The case in which m and $m+n$ are negative would require special examination, but the result in that case may be deduced from (96); change ψ, u into $-\psi$, $-u$, and ϕ into $2\pi-\phi$, we thus find

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{m\iota(\phi-\psi)\pm mu}}{\{\mu+\sqrt{\mu^2-1}\cos(\phi-\psi\pm\iota u)\}^{n+1}}\,d\phi=\frac{(-1)^{n+1}}{\Pi(m-n-1)}\,\frac{1}{\Pi(n+m)\,\Pi(n)}\,\mathrm{Q}_n^m(\mu)$$

when $m > n$, and is equal to zero when $m \overline{\overline{<}} n$ . . . . (97).

The results in (94), (96), (97) agree with those of HEINE,* the more general formulæ (93), (95) are not given by him.

45. Results such as those in Arts. 38, 43, 44, could be foreseen by a consideration of the fact that $(z+\alpha x+\beta y)^n$ satisfies LAPLACE's equation $\nabla^2\mathrm{V}=0$, provided α, β are any constants such that $\alpha^2+\beta^2=-1$; this holds for complex values of n, and when x, y, z are not restricted to be real. Let $\alpha=-\iota\cos(\psi\mp\iota u)$, $\beta=-\iota\sin(\psi\mp\iota u)$, then, since $z=r\mu$, $x=\iota r\sqrt{\mu^2-1}\cos\phi$, $y=\iota r\sqrt{\mu^2-1}\sin\phi$, we have

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* 'Kugelfunctionen,' vol. 1, p. 211.

$$(z + \alpha x + \beta y)^n = r^n \{\mu + \sqrt{\mu^2 - 1}\cos(\phi - \psi \pm \iota u)\}^n;$$

we should, therefore, expect that if $\{\mu + \sqrt{\mu^2 - 1}\cos(\phi - \psi \pm \iota u)\}^n$ is expanded in cosines and sines of multiples of ϕ, say $\Sigma(u_m \cos m\phi + v_m \sin m\phi)$, the coefficients u_m, v_m would be linear functions of the functions $P_n^m(\mu)$, $Q_n^m(\mu)$.

Let $w = \sqrt{\mu^2 - 1}\, e^{\pm\iota(\phi - \psi \pm \iota u)}$, we then find that

$$\{\mu + \sqrt{\mu^2 - 1}\cos(\phi - \psi \pm \iota u)\}^n = (2w)^{-n}\,(\mu + w - 1)^n(\mu + w + 1)^n.$$

If $u <$ log mod $\sqrt{\dfrac{\mu + 1}{\mu - 1}}$, one of the expressions $(\mu + w - 1)^n$, $(\mu + w + 1)^n$ can be expanded in positive powers, and the other in negative powers of w; if, however, $u >$ log mod $\sqrt{\dfrac{\mu + 1}{\mu - 1}}$, both expressions can be expanded in positive powers, or both in negative powers, according to the sign taken in $\pm\iota u$.

Case I.—$u <$ log mod $\sqrt{\dfrac{\mu + 1}{\mu - 1}}$.

In this case all the powers of w in the expansion are of positive or negative integral degree, thus

$$\{\mu + \sqrt{\mu^2 - 1}\cos(\phi - \psi \pm \iota u)\}^n = \sum_{m=0}^{\infty} u_m \cos m\phi + v_m \sin m\phi,$$

where m has all positive integral values.

We have

$$u_m = \frac{1}{\pi}\int_0^{2\pi}\{\mu + \sqrt{\mu^2 - 1}\cos(\phi - \psi \pm \iota u)\}^n \cos m\phi$$

$$= 2P_n^m(\mu)\frac{\Pi(n)}{\Pi(n + m)}\cos m(\psi \mp \iota u), \qquad \text{by (81)},$$

except that

$$u_0 = P_n(\mu),$$

also

$$v_m = \frac{1}{\pi}\int_0^{2\pi}\{\mu + \sqrt{\mu^2 - 1}\cos(\phi - \psi \pm \iota u)\}^n \sin m\phi$$

$$= 2P_n^m(\mu)\frac{\Pi(n)}{\Pi(n + m)}\sin m(\psi \mp \iota u),$$

hence

$$\{\mu + \sqrt{\mu^2 - 1}\cos(\phi - \psi \pm \iota u)\}^n$$

$$= P_n(\mu) + 2\sum_{m=1}^{\infty}\frac{\Pi(n)}{\Pi(n + m)}P_n^m(\mu)\cos m(\phi - \psi \pm \iota u) \quad (98),$$

this formula which holds for all real and complex values of n is a generalization of a well-known formula, namely the case in which n is a positive integer, in which case the series is a finite one, since $\mathrm{P}_n^m(\mu) = 0$, when m and n are positive integers such that $m > n$.

Case II. $u > \log \text{mod.} \sqrt{\frac{\mu+1}{\mu-1}}$.

In this case the expansion of $\{\mu + \sqrt{\mu^2-1}\cos(\phi - \psi - \iota u\}^n$ in powers of w consists of powers whose indices differ from n by a real integer; thus

$$\{\mu + \sqrt{\mu^2-1} \cos(\phi - \psi - \iota u)\}^n = \Sigma u_m e^{m(\phi-\psi-\iota u)\iota}$$

where m has the values $n, n-1, n-2, \ldots$

To determine u_m, multiply both sides of the equation by $e^{-m(\phi-\psi-\iota u)\iota}$; then, since $\int_0^{2\pi} e^{(m'-m)(\phi-\psi\pm\iota u)\iota}\,d\phi = 0$, when m, m' are different real integers, we have

$$u_m = \frac{1}{2\pi}\int_0^{2\pi} \{\mu + \sqrt{\mu^2-1} \cos(\phi - \psi - \iota u)\}^n e^{-m(\phi-\psi-\iota u)\iota}\,d\phi$$

$$= \frac{\Pi(n)}{\Pi(n+m)}\left\{\mathrm{P}_n^m(\mu) - \frac{2}{\pi}\cdot e^{-n\pi\iota}\sin n\pi\,.\,\mathrm{Q}_n^m(\mu)\right\}, \quad \ldots \text{ by (93).}$$

We have thus obtained the expansion

$$\{\mu + \sqrt{\mu^2-1} \cos(\phi - \psi - \iota u)\}^n$$
$$= \Sigma \frac{\Pi(n)}{\Pi(n+m)}\left\{\mathrm{P}_n^m(\mu) - \frac{2}{\pi} e^{-n\pi\iota}\sin n\pi\,.\,\mathrm{Q}_n^m(\mu)\right\} e^{m\iota(\phi-\psi-\iota u)} \quad \ldots \quad (99),$$

when m has the values $n, n-1, n-2, \ldots$ and the expansion holds for all real or complex values of n; in the special case in which n is a positive integer, we have

$$\{\mu + \sqrt{\mu^2-1} \cos(\phi - \psi - \iota u)\}^n = \sum_{m=0}^{m=n} \frac{\Pi(n)}{\Pi(n+m)}\mathrm{P}_n^m(\mu)\,.\,e^{\iota m(\phi-\psi-\iota u)} \quad . \quad (100).$$

When n is a negative integer, change it into $-n-1$; we thus find, on using the formula (97)

$$\{\mu + \sqrt{\mu^2-1} \cos(\phi - \psi - \iota u)\}^{-n-1}$$
$$= \frac{(-1)^{n+1}}{\Pi(n)}\Sigma\frac{1}{\Pi(m-n-1)\,\Pi(m+n)}\mathrm{Q}_n^m(\mu)\,e^{-\iota m(\phi-\psi-\iota u)}\,. \quad \ldots \quad (101),$$

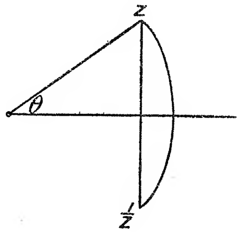
where m has the values $n+1, n+2, n+3, \ldots$

Generalization of DIRICHLET'S *and* MEHLER'S *Expressions for* $P_n(\cos\theta)$ *as a Definite Integral.*

46. It has been shown in Art. 33, that provided the real part of $m + \frac{1}{2}$ is positive,

$$P_n^{-m}(\mu) = \frac{1}{2^m} \cdot \frac{e^{-\iota\pi(m-\frac{1}{2})}}{\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}(\mu^2-1)^{-\frac{1}{2}m}\int_{\frac{1}{z}}^{z} h^{n-m}(1-2\mu h+h^2)^{m-\frac{1}{2}}\,dh,$$

let $\mu = \cos\theta + 0\,.\,\iota$, the line joining the points $z, \frac{1}{z}$ on the h-plane is perpendicular



to the real axis, and the path of integration may be taken to be a circular arc with centre at the origin; let $h = e^{\iota\phi}$, then remembering that the phase of $1 - 2\mu h + h^2$ increases from $2\pi - \theta$, at the lower limit to $2\pi + \theta$, at the upper one, we have

$$(1-2\mu h+h^2)^{m-\frac{1}{2}} = e^{2\pi(m-\frac{1}{2})\iota}\,e^{(m-\frac{1}{2})\iota\phi}\,(2\cos\phi - 2\cos\theta)^{m-\frac{1}{2}},$$

hence

$$\begin{aligned} P_n^{-m}(\cos\theta) &= e^{-\frac{1}{2}m\pi\iota}\,P_n^{-m}(\cos\theta + 0\,.\,\iota) \\ &= e^{-\frac{1}{2}m\pi\iota}\,\frac{1}{2^m}\,\frac{e^{-\iota\pi(m-\frac{1}{2})}}{\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\,.\,e^{-\frac{1}{2}m\pi\iota}\sin^{-m}\theta\,.\,e^{2\pi(m-\frac{1}{2})\iota} \\ &\qquad \int_{-\theta}^{\theta} e^{(n-m)\iota\phi}\,.\,e^{(m-\frac{1}{2})\iota\phi}(2\cos\phi-2\cos\theta)^{m-\frac{1}{2}}\,.\,\iota e^{\iota\phi}\,d\phi, \end{aligned}$$

or

$$P_n^{-m}(\cos\theta) = \frac{2}{2^m\,\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\sin^{-m}\theta\int_0^\theta \frac{\cos(n+\frac{1}{2})\phi}{(2\cos\phi-2\cos\theta)^{\frac{1}{2}-m}}\,d\phi \quad . \quad (102).$$

From Art. 11 we find

$$P_n^{-m}(\cos\theta) = \frac{\Pi(n-m)}{\Pi(n+m)}\left\{\cos m\pi\,.\,P_n^m(\cos\theta) - \frac{2}{\pi}\sin m\pi\,.\,Q_n^m(\cos\theta)\right\},$$

hence

$$\begin{aligned} &\frac{\Pi(n-m)}{\Pi(n+m)}\left\{\cos m\pi\,.\,P_n^m(\cos\theta) - \frac{2}{\pi}\sin m\pi\,.\,Q_n^m(\cos\theta)\right\} \\ &\qquad = \frac{2}{2^m\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\sin^{-m}\theta\,.\int_0^\theta \frac{\cos(n+\frac{1}{2})\phi}{(2\cos\phi-2\cos\theta)^{\frac{1}{2}-m}}\,d\phi \quad . \quad (103). \end{aligned}$$

A particular case of (102), or (103) is MEHLER'S form of one of DIRICHLET'S expressions for $P_n(\cos\theta)$,

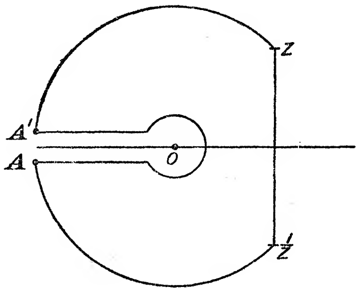
$$P_n(\cos\theta) = \frac{2}{\pi}\int_0^\theta \frac{\cos(n+\frac{1}{2})\phi}{2(\cos\phi - 2\cos\theta)^{\frac{1}{2}}}.$$

The formula (103) holds for all values of n and m, real or complex, provided the real part of $m+\frac{1}{2}$ is positive. When m is a real integer, we have for unrestricted values of n,

$$(-1)^m \frac{\Pi(n-m)}{\Pi(n+m)} P_n^m(\cos\theta) = \frac{2\sin^{-m}\theta}{2^m\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\int_0^\theta \frac{\cos(n+\frac{1}{2})\phi}{(2\cos\phi - 2\cos\theta)^{\frac{1}{2}-m}}d\phi \quad . \quad (104).$$

47. Next let us suppose the real parts of $n-m+1$, and of $m+\frac{1}{2}$ to be positive; the path of the integral

$$\int_{1/z}^{z} h^{n-m}(1-2\mu h+h^2)^{m-\frac{1}{2}}\,dh$$



can be taken as in the figure to consist of two circular arcs of unit radius, two straight portions along the real axis, and a circle of indefinitely small radius round the point $h=0$; under the above conditions as to m and n, the circle contributes nothing to the value of the integral.

In the integral taken along the arc joining the points $\frac{1}{z}$ and -1, the phase of $1-2\mu h+h^2$ is $3\pi-\phi$, where $h=e^{-\iota\phi}$; in the integral, from -1 to z, it is $\pi+\phi$, where $h=e^{\iota\phi}$; the two integrals together make up

$$\int_\theta^\pi \{e^{-(n-m)\iota\phi+(m-\frac{1}{2})(3\pi-\phi)\iota}(-\iota e^{-\iota\phi}) - e^{(n-m)\iota\phi+(m-\frac{1}{2})(\pi+\phi)\iota}(\iota e^{\iota\phi})\}(2\cos\theta - 2\cos\phi)^{m-\frac{1}{2}}\,d\phi,$$

or

$$e^{2\pi(m-\frac{1}{2})\iota}\int_\theta^\pi 2\iota\cos[(n+\tfrac{1}{2})\phi - (m+\tfrac{1}{2})\pi](2\cos\theta - 2\cos\phi)^{m-\frac{1}{2}}\,d\phi.$$

In the integrals from $h=1$ to $h=0$, the phase of $1-2\mu h+h^2$ is 2π; let $h=e^{-\iota\pi}e^{-v}$, for the lower path, and $h=e^{\iota\pi}e^{-v}$, for the upper path; these portions of the integral give us

$$-\int_0^\infty \{e^{-(n-m+1)\iota\pi} . e^{-(n+\frac{1}{2})v} . e^{2\pi\iota(m-\frac{1}{2})} - e^{(n-m+1)\iota\pi} . e^{-(n+\frac{1}{2})v} . e^{2\pi\iota(m-\frac{1}{2})}\} (2\cosh v + 2\cos\theta)^{m-\frac{1}{2}}\, dv,$$

or

$$-e^{(m-\frac{1}{2})2\pi\iota}\, 2\iota \sin(n-m)\pi \int_0^\infty \frac{e^{-(n+\frac{1}{2})v}}{(2\cosh v + 2\cos\theta)^{\frac{1}{2}-m}}\, dv$$

we thus obtain the formula

$$\begin{aligned} \mathrm{P}_n^{-m}(\cos\theta) &= \frac{\Pi(n-m)}{\Pi(n+m)} \left\{ \cos m\pi \,.\, \mathrm{P}_n^m(\cos\theta) - \frac{2}{\pi} \sin m\pi \,.\, \mathrm{Q}_n^m(\cos\theta) \right\} \\ &= \frac{2\sin^{-m}\theta}{2^m \Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})} \left\{ \int_\theta^\pi \frac{\cos[(n+\frac{1}{2})\phi - (m+\frac{1}{2})\pi]}{(2\cos\theta - 2\cos\phi)^{\frac{1}{2}-m}}\, d\phi \right. \\ &\qquad \left. + \cos(n+\tfrac{1}{2}-m)\pi \int_0^\infty \frac{e^{-(n+\frac{1}{2})v}}{(2\cosh v + 2\cos\theta)^{\frac{1}{2}-m}} \right\} \quad . \quad (105), \end{aligned}$$

which holds provided the real parts of $m + \frac{1}{2}$, $n - m + 1$ are positive. If $n - m$ is a positive real integer this becomes

$$\begin{aligned} &\frac{\Pi(n-m)}{\Pi(n+m)} \left\{ \cos m\pi \,.\, \mathrm{P}_n^m(\cos\theta) - \frac{2}{\pi} \sin m\pi \,.\, \mathrm{Q}_n^m(\cos\theta) \right\} \\ &\qquad = \frac{2\sin^{-m}\theta}{2^m \Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})} \int_\theta^\pi \frac{\cos[(n+\frac{1}{2})\phi - (m+\frac{1}{2})\pi]}{(2\cos\theta - 2\cos\phi)^{\frac{1}{2}-m}}\, d\phi \quad . \quad . \quad (106). \end{aligned}$$

When m and n are both positive integers, and $n \geqq m$, we obtain

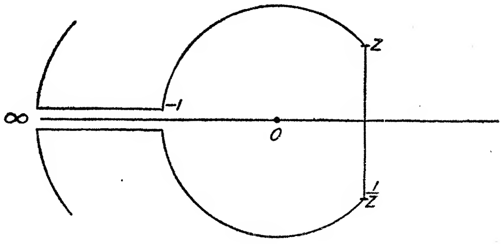
$$\frac{\Pi(n-m)}{\Pi(n+m)} \mathrm{P}_n^m(\cos\theta) = \frac{2\sin^{-m}\theta}{2^m \Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})} \int_\theta^\pi \frac{\sin(n+\frac{1}{2})\phi}{(2\cos\theta - 2\cos\phi)^{\frac{1}{2}-m}}\, d\phi \quad . \quad (107),$$

which becomes, when $m = 0$,

$$\mathrm{P}_n(\cos\theta) = \frac{2}{\pi} \int_\theta^\pi \frac{\sin(n+\frac{1}{2})\phi}{(2\cos\theta - 2\cos\phi)^{\frac{1}{2}}}\, d\phi \quad . \quad . \quad . \quad . \quad . \quad (108),$$

which is the second expression given by MEHLER for $\mathrm{P}_n(\cos\theta)$. The formulæ (105), (106), (107) are therefore generalizations of the known formulæ of MEHLER and DIRICHLET.

48. Next suppose the condition that the real part of $n - m + 1$ is positive does not necessarily hold, but that the real part of $n + m$ is negative, and that of $m + \frac{1}{2}$ is positive; we may replace part of the path of integration in the last Art. by straight paths from -1 to $-\infty$ along the real axis, and a circle of infinite radius. From $\frac{1}{z}$ to -1, the phase of $1 - 2\mu h + h^2$ is $\pi - \phi$, where ϕ is initially equal to θ; from -1 to z, the phase of $1 - 2\mu h + h^2$ is $3\pi + \phi$, where ϕ is equal to θ at the



point z. The part of the integral for $\mathrm{P}_n^{-m}(\cos\theta)$ which consists of integrations along the two finite circular axes is

$$\frac{1}{2^m}\cdot\frac{e^{-\iota\pi(m-\frac{1}{2})}}{\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\,e^{-m\pi\iota}\sin^{-m}\theta\int_\theta^\pi\Big\{e^{-(n-m)\iota\phi+(m-\frac{1}{2})(\pi-\phi)\iota}\left(-\iota e^{-\iota\phi}\right)$$
$$-\,e^{(n-m)\iota\phi+(m-\frac{1}{2})(3\pi+\phi)\iota}\left(\iota e^{\iota\phi}\right)\Big\}\,(2\cos\theta-2\cos\phi)^{m-\frac{1}{2}}\,d\phi$$

or

$$\frac{1}{2^m\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\sin^{-m}\theta\int_\theta^\pi 2\cos\{(m+\tfrac{1}{2})\pi+(n+\tfrac{1}{2})\phi\}\,(2\cos\theta-2\cos\phi)^{m-\frac{1}{2}}\,d\phi.$$

The part of the integral which is taken along the circle of infinite radius is zero, and the part taken along the real axis is

$$\frac{1}{2^m}\cdot\frac{e^{-\iota\pi(m-\frac{1}{2})}}{\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\,e^{-m\pi\iota}\sin^{-m}\theta\int_0^\infty\Big\{e^{-\iota\pi(n-m+1)+(n+\frac{1}{2})v}$$
$$-\,e^{\iota\pi(n-m+1)+(n+\frac{1}{2})v+(m-\frac{1}{2})4\pi\iota}\Big\}\,(2\cos\theta+2\cosh v)^{m-\frac{1}{2}}\,dv$$

or

$$\frac{1}{2^m\,\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\sin^{-m}\theta\int_0^\infty 2e^{(n+\frac{1}{2})v}\cos(n+\tfrac{1}{2}+m)\pi\,(2\cos\theta+2\cosh v)^{m-\frac{1}{2}}\,dv;$$

we thus obtain the formula

$$\mathrm{P}_n^{-m}(\cos\theta)=\frac{\Pi(n-m)}{\Pi(n+m)}\left\{\cos m\pi\,.\,\mathrm{P}_n^m(\cos\theta)-\frac{2}{\pi}\sin m\pi\,.\,\mathrm{Q}_n^m(\cos\theta)\right\}$$
$$=\frac{1}{2^m\,\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\cdot\frac{2}{\sin^m\theta}\left\{\int_0^\pi\cos(\overline{n+\tfrac{1}{2}}\phi+\overline{m+\tfrac{1}{2}}\pi)\,(2\cos\theta-2\cos\phi)^{m-\frac{1}{2}}\,d\phi\right.$$
$$\left.+\int_0^\infty e^{(n+\frac{1}{2})v}\cos(n+\tfrac{1}{2}+m)\pi\,.\,(2\cos\theta+2\cosh v)^{m-\frac{1}{2}}\,dv\right\}\quad(109)$$

which holds, provided the real part of $n+m$ is negative, and that of $m+\frac{1}{2}$ is positive.

When the real part of n is between 0 and -1, and the real part of m is between

$\frac{1}{2}$ and $-\frac{1}{2}$, both the formulæ (109) and (105) hold. In the special case $m=0$, we find by adding the two expressions in (105) and (109),

$$P_n(\cos\theta) = \frac{2}{\pi}\cos(n+\tfrac{1}{2})\pi\int_0^\infty \frac{\cosh(n+\frac{1}{2})v}{(2\cos\theta+2\cosh v)^{\frac{1}{2}}}\,dv \quad . \quad . \quad . \quad (110)$$

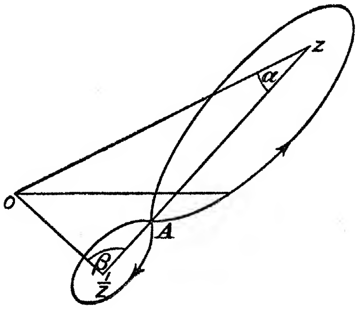
when the real part of n is between 0 and -1.

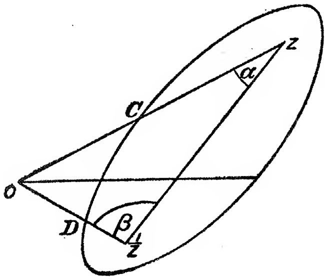
A definite integral form for $P_n(\mu)$, *when the real part of* n *is between* 0 *and* -1.

49. Taking the formula

$$P_n^m(\mu) = \frac{1}{2\pi\iota}\cdot 2^m\frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\int^{(z+,\,1/z-)}\frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}}\,dh,$$

we see that, provided m is half a real integer, and also $\frac{1}{2}-m$ is positive, the path may be replaced by one which consists of a single curve enclosing both the points $z, \frac{1}{z}$.

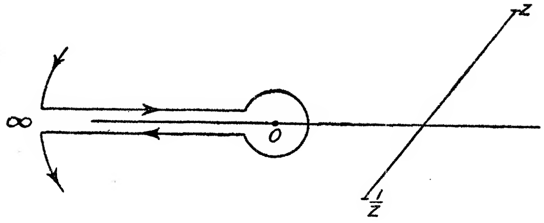




In the first figure the initial phases at A are $2\pi-\beta$, for $1-hz$, and $-(2\pi-\alpha)$ for $1-\frac{h}{z}$. In the second figure the phase of $1-\frac{h}{z}$ is zero at C, and that of $1-hz$ is 2π at D. The formula becomes

$$P_n^m(\mu) = \frac{1}{2\pi\iota}\,2^m\frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\int^{(z+,\,1/z+)}\frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}}\,dh.$$

Now suppose the real part of $n-m$ is negative, and that of $n+m+1$ is positive; we may replace the path by one round a circle of infinite radius, straight paths along



the real axis, and a circle round the point O; the circular paths contribute nothing to the value of the integral, and we have

$$\mathrm{P}_n^m(\mu) = \frac{1}{2\pi\iota}\, 2^m \frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})} (\mu^2-1)^{\frac{1}{2}m} \int_{-\infty}^{\infty} - \{e^{(n+m+1)\iota\pi - 2\iota\pi(m+\frac{1}{2})}$$

$$- e^{-(n+m+1)\iota\pi + 2\iota\pi(m+\frac{1}{2})}\} e^{(n+\frac{1}{2})u} (2\cosh u + 2\mu)^{-m-\frac{1}{2}}\, du$$

$$= -\frac{2^{m+1}}{\pi} \cdot \frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})} \sin(n-m)\pi \,.\, \int_0^{\infty} \cosh(n+\tfrac{1}{2})u \,.\, (2\cosh u + 2\mu)^{-m-\frac{1}{2}}\, du,$$

this holds for all values of μ of which the real part is positive, provided m is half an integer, and is less than $\frac{1}{2}$, also provided the real part of $n - m$ is negative and of $n + m + 1$ is positive; the only value of m which satisfies these conditions is $m = 0$; we thus obtain the formula

$$\mathrm{P}_n(\mu) = \frac{2}{\pi} \cos(n+\tfrac{1}{2})\pi \int_0^{\infty} \frac{\cosh(n+\frac{1}{2})u}{(2\cosh u + 2\mu)^{\frac{1}{2}}}\, du \quad . \quad . \quad . \quad . \qquad (111),$$

which holds, provided the real part of n is between 0 and -1, and that of μ is positive.

Definite Integral Expressions for $\mathrm{P}_n^m(\mu)$, *when* μ *is real and greater than unity.*

50. In the formula

$$\mathrm{P}_n^{-m}(\mu) = \frac{1}{2^m\, \Pi(-\frac{1}{2})\, \Pi(m-\frac{1}{2})} e^{-\iota\pi(m-\frac{1}{2})} (\mu^2-1)^{-\frac{1}{2}m} \int_{1/z}^{z} h^{n-m} (1 - 2\mu h + h^2)^{m-\frac{1}{2}}\, dh$$

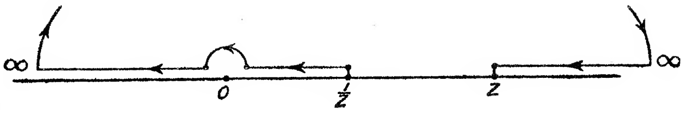
where the real part of $m + \frac{1}{2}$ is positive; when μ is real and greater than unity, put $\mu = \cosh\psi$, then $z = e^{\psi}$, $\frac{1}{z} = e^{-\psi}$, thus putting $h = e^{\phi}$, we obtain the formula

$$P_n^{-m}(\cosh\psi)$$

$$= \frac{1}{2^m\Pi(-\tfrac{1}{2})\Pi(m-\tfrac{1}{2})}\sinh^{-m}\psi\int_0^{\psi} 2\cosh(n+\tfrac{1}{2})\phi\,(2\cosh\psi - 2\cosh\phi)^{m-\frac{1}{2}}\,d\phi \quad . \quad (112)$$

where the real part of $m+\frac{1}{2}$ is positive, and in particular

$$P_n(\cosh\psi) = \frac{2}{\pi}\int_0^{\psi}\frac{\cosh(n+\frac{1}{2})\phi}{\sqrt{2\cosh\psi - 2\cosh\phi}}\,d\phi \quad . \quad . \quad . \quad . \quad (113).$$



The path joining the points z, $1/z$ can be placed as in the figure, along the real axis from $1/z$ to $-\infty$, except for a small semicircle round the point 0, then a semicircle of infinite radius, and lastly a straight path along the real axis from $+\infty$ to z. If the real part of m lies between $\frac{1}{2}$ and $-\frac{1}{2}$, and if the real part of $n-m+1$ is positive, and if $n+m$ is negative, the straight portions of the path are the only ones which contribute anything to the value of the integral; in this way we find that under the conditions just specified.

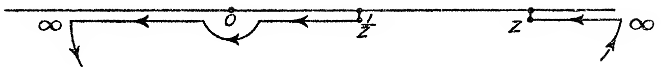
$$P_n^{-m}(\cosh\psi) = \frac{\sinh^{-m}\psi}{2^m\Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})}\left\{\int_{\psi}^{\infty} -2\iota\sinh(\overline{n+\tfrac{1}{2}}\,\phi - \iota m\pi)\,(2\cosh\phi - 2\cosh\psi)^{m-\frac{1}{2}}\,d\phi\right.$$

$$\left.+\int_0^{\infty} e^{(n+\frac{1}{2})\phi + (n+\frac{1}{2})\iota\pi}(2\cosh\phi + 2\cosh\psi)^{m-\frac{1}{2}}\,d\phi\right\}.$$

In a similar manner, we can prove, by taking the semicircles below the real axis that under the same conditions, $P_n^{-m}(\cosh\psi)$ is given by the formula

$$P_n^{-m}(\cosh\psi) = \frac{\sinh^{-m}\psi}{2^m\Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})}\left\{\int_{\psi}^{\infty} 2\iota\sinh(\overline{n+\tfrac{1}{2}}\,\phi + \iota m\pi)\,(2\cosh\phi - 2\cosh\psi)^{m-\frac{1}{2}}\,d\phi\right.$$

$$\left.+\int_0^{\infty} e^{(n+\frac{1}{2})\phi - (n+\frac{1}{2})\iota\pi}(2\cosh\phi + 2\cosh\psi)^{m-\frac{1}{2}}\,d\phi\right\}.$$



Multiply the two formulæ by $e^{-(n+\frac{1}{2})\iota\pi}$, $e^{(n+\frac{1}{2})\iota\pi}$, and subtract them, we then have the formula

$$\mathrm{P}_n^{-m}(\cosh\psi) = \frac{\sinh^{-m}\psi}{2^m\Pi(-\tfrac{1}{2})\,\Pi(m-\tfrac{1}{2})}\operatorname{cosec}(n+\tfrac{1}{2})\,\pi\int_\psi^\infty\{e^{(n+\frac{1}{2})\iota\pi}\sinh(\overline{n+\tfrac{1}{2}}\phi+\iota m\pi)$$

$$+\,e^{-(n+\frac{1}{2})\iota\pi}\sinh(\overline{n+\tfrac{1}{2}}\phi-im\pi)\}\,(2\cosh\phi-2\cosh\psi)^{m-\frac{1}{2}}\,d\phi \quad (114),$$

where the real part of n is between 0 and -1; and the real part of m is between $\pm\frac{1}{2}$. If $m = 0$, we have

$$\mathrm{P}_u(\cosh\psi) = \frac{2}{\pi}\cot(n+\tfrac{1}{2})\,\pi\int_\psi^\infty\frac{\sinh(n+\frac{1}{2})\,\phi}{\sqrt{2\cosh\phi-2\cosh\psi}}\,d\phi \quad . \quad . \quad (115).$$

Definite Integral Formulæ for $Q_n^m(\cos\theta)$, *under Special Conditions.*

51. When the real parts of $n+m+1$, $\frac{1}{2}-m$ are positive, we have

$$\mathrm{Q}_n^m(\cos\theta+0\,.\,\iota) = e^{m\pi\iota}\,.\,2^m\Pi(m-\tfrac{1}{2})\,\Pi(-\tfrac{1}{2})\,\frac{\cos m\pi}{\pi}\,.\,e^{\frac{m\pi\iota}{2}}\,.\,\sin^m\theta\int_0^{1/z}\frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}}\,dh \quad . \quad (43),$$

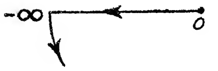
take the path of integration to be from 0 to 1, along the real axis, then from 1 to $\frac{1}{z}$ along an arc of a circle of unit radius with its centre at the origin; along the straight path, $1-2\mu h+h^2$ has the phase zero, and along the circular arc it has the same phase as h, hence, writing in the first part of the integral $h = e^{-u}$ and in the second part $h = e^{-\iota\phi}$,

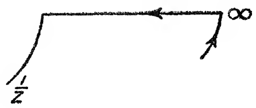
$$\mathrm{Q}_n^m(\cos\theta+0\,.\,\iota)$$

$$= e^{\frac{3}{2}m\pi\iota}\,.\,2^m\,.\,\Pi(m-\tfrac{1}{2})\,\Pi(-\tfrac{1}{2})\,\frac{\cos m\pi}{\pi}\,.\,\sin^m\theta\left\{\int_0^\infty\frac{e^{-(n+\frac{1}{2})u}}{(2\cosh u-2\cos\theta)^{m+\frac{1}{2}}}\,du\right.$$

$$\left.-\int_0^\theta\frac{\iota e^{-(n+\frac{1}{2})\iota\phi}}{(2\cos\phi-2\cos\theta)^{m+\frac{1}{2}}}\,d\phi\right\}.$$

Next take the path to be from 0 to $-\infty$ along the real axis, along a semicircle of infinite radius to $+\infty$, from $+\infty$ along the real axis to 1, and from 1 along a circular arc whose centre is the origin to the point $\frac{1}{z}$. If the real part of $n - m$ is negative, the part of the integral taken along the infinite semicircle is zero; we have





then, writing $h = e^{-\iota\pi}.e^{u}$, in the first integral, $h = e^{u}$, in the second integral, and $h = e^{-\iota\phi}$, in the third integral,

$$
\begin{aligned}
&\mathrm{Q}_n^m(\cos\theta + 0\,.\,\iota)\\
&\quad = e^{\frac{3}{2}m\pi\iota}\,.\,2^m\,.\,\Pi\left(m - \tfrac{1}{2}\right)\Pi\left(-\tfrac{1}{2}\right)\frac{\cos m\pi}{\pi}\sin^m\theta\left\{\int_{-\infty}^{\infty} e^{-(n+m+1)\iota\pi}\frac{e^{(n+\frac{1}{2})u}}{(2\cosh u + 2\cos\theta)^{m+\frac{1}{2}}}\,du\right.\\
&\qquad\qquad \left. - \int_0^{\infty}\frac{1}{e^{(m+\frac{1}{2})2\pi\iota}}\frac{e^{(n+\frac{1}{2})u}}{(2\cosh u - 2\cos\theta)^{m+\frac{1}{2}}}\,du - \int_0^{\theta}\frac{\iota e^{-(n+\frac{1}{2})\iota\phi}}{e^{(m+\frac{1}{2})2\pi\iota}(2\cos\phi - 2\cos\theta)^{m+\frac{1}{2}}}\,d\phi\right\}.
\end{aligned}
$$

If the real part of m lies between $\pm\frac{1}{2}$, and if the real parts of $n + m + 1$, $m - n$ are positive, both the formulæ we have found for $\mathrm{Q}_n^m(\cos\theta + 0\,.\,\iota)$ hold. Multiply the first expression by $e^{-m\pi\iota}$, and the second by $e^{m\pi\iota}$, and then add; we find

$$
\begin{aligned}
&2\cos m\pi\,.\,\mathrm{Q}_n^m(\cos\theta + 0\,.\,\iota)\\
&\quad = e^{\frac{1}{2}m\pi\iota}\,.\,2^m\,.\,\Pi\left(m - \tfrac{1}{2}\right)\Pi\left(-\tfrac{1}{2}\right)\frac{\cos m\pi}{\pi}\,.\,\sin^m\theta\left\{\int_0^{\infty}\frac{2\cosh\left(n + \frac{1}{2}\right)u}{(2\cosh u - 2\cos\theta)^{m+\frac{1}{2}}}\,du\right.\\
&\qquad\qquad \left. + e^{-(n-m+1)\iota\pi}\int_0^{\infty}\frac{2\cosh\left(n + \frac{1}{2}\right)u}{(2\cosh u + 2\cos\theta)^{m+\frac{1}{2}}}\,du\right\} \quad . \quad (116),
\end{aligned}
$$

which holds, provided the real part of m lies between $\pm\frac{1}{2}$, and those of $n + m + 1$, $m - n$ are positive.

If $m = 0$, we have

$$
\mathrm{Q}_n(\cos\theta + 0\,.\,\iota) = \int_0^{\infty}\frac{\cosh\left(n + \frac{1}{2}\right)u}{(2\cosh u - 2\cosh\theta)^{\frac{1}{2}}}\,du + e^{-(n+1)\iota\pi}\int_0^{\infty}\frac{\cosh\left(n + \frac{1}{2}\right)u}{(2\cosh u + 2\cos\theta)^{\frac{1}{2}}}\,du \qquad (117),
$$

which holds provided the real part of n is between 0 and -1.

It is to be remembered that $\mathrm{Q}_n(\cos\theta + 0\,.\,\iota) = \mathrm{Q}_n(\cos\theta) - \frac{\iota\pi}{2}\mathrm{P}_n(\cos\theta)$.

Formula for $Q_n^m(\cosh\psi)$ *under special conditions.*

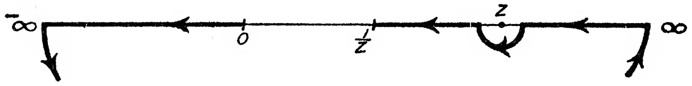
52. When μ is real and greater than unity, let $\mu = \cosh\psi$, we then have, provided the real parts of $n + m + 1$, $\frac{1}{2} - m$ are positive,

$$Q_n^m(\cosh\psi) = e^{m\pi\iota} \,.\, 2^m \,.\, \Pi\left(m - \tfrac{1}{2}\right)\Pi\left(-\tfrac{1}{2}\right)\frac{\cos m\pi}{\pi}\sinh^m\psi\int_0^{1/z}\frac{h^{n+m}}{(1 - 2\mu h + h^2)^{m+\frac{1}{2}}}\,dh.$$

Let $h = e^{-u}$, we have then, taking the path along the real axis,

$$Q_n^m(\cosh\psi) = e^{m\pi\iota} \,.\, 2^m \,.\, \Pi\left(m - \tfrac{1}{2}\right)\Pi\left(-\tfrac{1}{2}\right)\frac{\cos m\pi}{\pi}\sinh^m\psi\int_\psi^\infty\frac{e^{-(n+\frac{1}{2})u}}{(2\cosh u - 2\cosh\psi)^{m+\frac{1}{2}}}\,du, \quad (118).$$

If we take the path to be from 0 to $-\infty$ along the real axis, along an infinite semicircle from $-\infty$ to $+\infty$, along a straight path from ∞ to $\frac{1}{z}$, avoiding the



point z by describing a small semicircle; the integrals along the semicircles vanish provided the real part of $n - m$ is negative, we then have

$$Q_n^m(\cosh\psi) = e^{m\pi\iota} \,.\, 2^m \,.\, \Pi\left(m - \tfrac{1}{2}\right)\Pi\left(-\tfrac{1}{2}\right)\frac{\cos m\pi}{\pi}\sinh^m\psi\left\{\int_{-\infty}^{\infty}\frac{e^{-(n+m+1)\iota\pi} \,.\, e^{(n+\frac{1}{2})u}}{(2\cosh\psi + 2\cosh u)^{m+\frac{1}{2}}}\,du\right.$$
$$\left. - \int_\psi^\infty\frac{e^{(n+\frac{1}{2})u}}{e^{(2m+1)\iota\pi}(2\cosh u - 2\cosh\psi)^{m+\frac{1}{2}}}\,du - \int_0^\psi\frac{e^{-(n+\frac{1}{2})u}}{e^{(m+\frac{1}{2})\iota\pi}(2\cosh\psi - 2\cosh u)^{m+\frac{1}{2}}}\,du\right\}.$$

In a similar manner, by taking the semicircles above the real axis, we can show that

$$Q_n^m(\cosh\psi) = e^{m\pi\iota} \,.\, 2^m\Pi\left(m - \tfrac{1}{2}\right)\Pi\left(-\tfrac{1}{2}\right)\frac{\cos m\pi}{\pi}\sinh^m\psi\left\{\int_{-\infty}^{\infty}\frac{e^{(n+m+1)\iota\pi} \,.\, e^{(n+\frac{1}{2})u}}{(2\cosh\psi + 2\cosh u)^{m+\frac{1}{2}}}\,du\right.$$
$$\left. - \int_\psi^\infty\frac{e^{(n+\frac{1}{2})u}}{e^{-(2m+1)\iota\pi}(2\cosh u - 2\cosh\psi)^{m+\frac{1}{2}}}\,du - \int_0^\psi\frac{e^{-(n+\frac{1}{2})u}}{e^{-(m+\frac{1}{2})\iota\pi}(2\cosh\psi - 2\cosh u)^{m+\frac{1}{2}}}\,du\right\}.$$

Multiplying the first expression by $e^{(n+m+1)\iota\pi}$, and the second by $e^{-(n+m+1)\iota\pi}$, and subtracting, we then have

$$
\begin{aligned}
&Q_n^m(\cosh\psi)\sin(n+m)\pi \\
&= e^{m\pi\iota}.2^m.\Pi(m-\tfrac{1}{2})\Pi(-\tfrac{1}{2})\frac{\cos m\pi}{\pi}\sinh^m\psi\left\{\sin(n-m)\pi\int_\psi^\infty\frac{e^{(n+\frac{1}{2})u}}{(2\cosh u-2\cosh\psi)^{m+\frac{1}{2}}}du\right. \\
&\qquad\qquad\left.-\sin(n+\tfrac{1}{2})\pi\int_0^\psi\frac{e^{-(n+\frac{1}{2})u}}{(2\cosh\psi-2\cosh u)^{m+\frac{1}{2}}}du\right\};
\end{aligned}
$$

where the real part of m is less than $\frac{1}{2}$, and the real parts of $n+m+1$, $m-n$ are positive.

Put $m=0$, we have then

$$
Q_n(\cosh\psi)=\int_\psi^\infty\frac{e^{(n+\frac{1}{2})u}}{(2\cosh u-2\cosh\psi)^{\frac{1}{2}}}du-\cot n\pi\int_0^\psi\frac{e^{-(n+\frac{1}{2})u}}{(2\cosh\psi-2\cosh u)^{\frac{1}{2}}}du, \quad (119)
$$

where the real part of n lies between 0 and -1.

Expressions for $Q_n^m(\cosh\psi)$, *when* $n-\frac{1}{2}$ *is a real integer.*

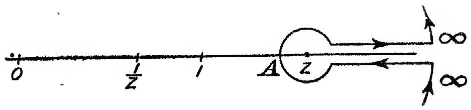
53. When $n-\frac{1}{2}$ is a real integer, the formula

$$
Q_n^m(\mu)=\iota e^{(m-n)\pi\iota}.2^m.\frac{\Pi(m-\frac{1}{2})\Pi(-\frac{1}{2})}{4\pi\sin(n+m)\pi}(\mu^2-1)^{\frac{1}{2}m}\int^{(1/z,\,0,\,1/z-,\,0-)}\frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}}dh
$$

may be replaced by

$$
e^{2m\pi\iota}.2^m.\frac{\Pi(m-\frac{1}{2})\Pi(-\frac{1}{2})}{2\pi}(\mu^2-1)^{\frac{1}{2}m}\int^{(1/z+,\,0+)}\frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}}dh,
$$

the path may, as in the figure, consist of a circle of infinite radius, straight paths along the real axis from ∞ to z, and a small circle round the point z.



If the real parts of $m-n$, $\frac{1}{2}-m$ are positive, the only effective parts of the integral are those along the real axis. The phase of $1-2\mu h+h^2$, at A is π; we thus find,

$$
\begin{aligned}
&Q_n^m(\cosh\psi) \\
&= e^{2m\pi\iota}.2^m.\frac{\Pi(m-\frac{1}{2})\Pi(-\frac{1}{2})}{2\pi}\sinh^m\psi\left\{\int_\psi^\infty\frac{e^{(n+\frac{1}{2})u}}{(2\cosh u-2\cosh\psi)^{m+\frac{1}{2}}}du\right. \\
&\qquad\qquad\left.-\int_\psi^\infty\frac{e^{(n-m)2\pi\iota}.e^{(n+\frac{1}{2})u}}{(2\cosh u-2\cosh\psi)^{m+\frac{1}{2}}}du\right\}, \\
&= e^{(m+n)\pi\iota}.2\iota\sin(m-n)\pi.\frac{2^m\Pi(m-\frac{1}{2})\Pi(-\frac{1}{2})}{2\pi}\sinh^m\psi\int_\psi^\infty\frac{e^{(n+\frac{1}{2})u}}{(2\cosh u-2\cosh\psi)^{m+\frac{1}{2}}}du,
\end{aligned}
$$

or

$$\mathrm{Q}_n^m(\cosh\psi)$$

$$= \frac{\cos m\pi\, 2^m\, \Pi(m-\tfrac{1}{2})\, \Pi(-\tfrac{1}{2})}{\pi} \sinh^m\psi \int_\psi^\infty \frac{e^{(n+\frac{1}{2})u}}{(2\cosh u - 2\cosh\psi)^{m+\frac{1}{2}}}\, du \quad . \quad . \quad . \quad . \quad (120),$$

where $n - \frac{1}{2}$ is a real integer, and the real parts of $m - n$, $\frac{1}{2} - m$ are positive.

If $m = 0$, we have

$$\mathrm{Q}_n(\cosh\psi) = \int_\psi^\infty \frac{e^{(n+\frac{1}{2})u}}{(2\cosh u - 2\cosh\psi)^{\frac{1}{2}}}\, du \quad . \quad . \quad . \quad . \quad . \quad (121),$$

where $n - \frac{1}{2}$ is a negative real integer.

For all values of m and n such that $n - \frac{1}{2}$ is a real integer, the path may be taken to be a circle of radius unity with the origin as centre; we obtain on putting $h = e^{\iota\phi}$, since $h^2 - 2\mu h + 1 = he^{\iota\pi}(2\cosh\psi - 2\cos\phi)$,

$$\mathrm{Q}_n^m(\mu) = e^{2m\pi\iota} \,.\, 2^m \,.\, \frac{\Pi(m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{2\pi} \sinh^m\psi \int_{-\pi}^{\pi} \frac{e^{(n-\frac{1}{2})\iota\phi} \,.\, \iota \,.\, e^{\iota\phi}}{e^{(m+\frac{1}{2})\iota\pi}(2\cosh\psi - 2\cos\phi)^{m+\frac{1}{2}}}\, d\phi,$$

or

$$\mathrm{Q}_n^m(\mu) = \iota e^{(m-\frac{1}{2})\iota\pi} \,.\, \frac{2^m\, \Pi(m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{\pi} \sinh^m\psi \int_0^\pi \frac{\cos(n+\frac{1}{2})\phi}{(2\cosh\psi - 2\cos\phi)^{m+\frac{1}{2}}}\, d\phi \quad (122).$$

Recurrent Relations for Successive Values of n, m in $\mathrm{P}_n^m(\mu)$, $\mathrm{Q}_n^m(\mu)$.

54. Denote the integral $\int \frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}}\, dh$, by $\mathrm{U}(n, m)$, the integral being taken along any closed path, that is, one in which after completion the integrand returns to its initial value.

We find

$$\frac{d\mathrm{U}(n, m)}{d\mu} = (2m+1) \int \frac{h^{n+m+1}}{(1-2\mu h+h^2)^{m+\frac{3}{2}}}\, dh = (2m+1)\, \mathrm{U}(n, m+1);$$

also

$$\frac{d}{dh} \cdot \frac{\mu - h}{(1-2\mu h+h^2)^{m+\frac{1}{2}}} = \frac{2m}{(1-2\mu h+h^2)^{m+\frac{1}{2}}} + (2m+1) \frac{\mu^2-1}{(1-2\mu h+h^2)^{m+\frac{3}{2}}} \,.$$

Hence

$$(\mu^2-1)\frac{d\mathrm{U}(n, m)}{d\mu} = \int h^{n+m+1} \,.\, \left\{ \frac{-2m}{(1-2\mu h+h^2)^{m+\frac{1}{2}}} + \frac{d}{dh} \cdot \frac{\mu - h}{(1-2\mu h+h^2)^{m+\frac{1}{2}}} \right\} dh$$

$$= -2m\mathrm{U}(n+1, m) - \int \frac{\mu - h}{(1-2\mu h+h^2)^{m+\frac{1}{2}}} (n+m+1)\, h^{n+m}\, dh$$

$$= -2m\mathrm{U}(n+1, m) - (n+m+1)\{\mu\mathrm{U}(n, m) - \mathrm{U}(n+1, m)\},$$

or,

$$(\mu^2 - 1)\frac{d\mathrm{U}(n, m)}{d\mu} = (n - m + 1)\,\mathrm{U}(n + 1, m) - (n + m + 1)\,\mu\mathrm{U}(n, m).$$

Referring to the formulæ (40), (50) for $\mathrm{Q}_n^m(\mu)$, $\mathrm{P}_n^m(\mu)$, we see that by choosing specified closed paths for the integration in $\mathrm{U}(n, m)$, each of the functions is of the form $\mathrm{C}_m(\mu^2 - 1)^{\frac{1}{2}m}\mathrm{U}(n, m)$; we thus obtain the formulæ

$$\left.\begin{aligned}(\mu^2 - 1)\frac{d\mathrm{P}_n^m(\mu)}{d\mu} &= (n - m + 1)\,\mathrm{P}_{n+1}^m(\mu) - (n + 1)\,\mu\mathrm{P}_n^m(\mu)\\ (\mu^2 - 1)\frac{d\mathrm{Q}_n^m(\mu)}{d\mu} &= (n - m + 1)\,\mathrm{Q}_{n+1}^m(\mu) - (n + 1)\,\mu\mathrm{Q}_n^m(\mu)\end{aligned}\right\} \quad . \quad (123).$$

Next let $\mathrm{V}(n, m) = \mathrm{U}(-n - 1, m)$, we have then by changing n into $-n - 1$ in the relation which has been found above for U,

$$(\mu^2 - 1)\frac{d\mathrm{V}(n, m)}{d\mu} = -(n + m)\,\mathrm{V}(n - 1, m) + n\mu\,\mathrm{V}(n, m);$$

special cases of this relation are

$$\left.\begin{aligned}(\mu^2 - 1)\frac{d\mathrm{P}_n^m(\mu)}{d\mu} &= n\,\mu\mathrm{P}_n^m(\mu) - (n + m)\,\mathrm{P}_{n-1}^m(\mu)\\ (\mu^2 - 1)\frac{d\mathrm{Q}_n^m(\mu)}{d\mu} &= n\,\mu\mathrm{Q}_n^m(\mu) - (n + m)\,\mathrm{Q}_{n-1}^m(\mu)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (124),$$

from (123), (124), we have at once

$$\left.\begin{aligned}(2n + 1)\,\mu\mathrm{P}_n^m(\mu) - (n - m + 1)\,\mathrm{P}_{n+1}^m(\mu) - (n + m)\,\mathrm{P}_{n-1}^m(\mu) = 0\\ (2n + 1)\,\mu\mathrm{Q}_n^m(\mu) - (n - m + 1)\,\mathrm{Q}_{n+1}^m(\mu) - (n + m)\,\mathrm{Q}_{n-1}^m(\mu) = 0\end{aligned}\right\} \quad . \quad (125),$$

these recurrent relations between the functions for different values of n hold for general complex values of m and n.

55. It has been remarked in Art. 1, that W, which is equivalent to $\mathrm{U}(n, m)$, satisfies the differential equation

$$(1 - \mu^2)\frac{d^2\mathrm{U}(n, m)}{d\mu^2} - 2(m + 1)\,\mu\frac{d\mathrm{U}(n, m)}{d\mu} + (n - m)(n + m + 1)\,\mathrm{U}(n, m) = 0,$$

now

$$\frac{d\mathrm{U}(n, m)}{d\mu} = (2m + 1)\,\mathrm{U}(n, m + 1), \quad \frac{d^2\mathrm{U}(n, m)}{d\mu^2} = (2m + 1)(2m + 3)\,\mathrm{U}(n, m + 2)$$

thus

$$\begin{aligned}(1 - \mu^2)(2m + 1)(2m + 3)\,\mathrm{U}(n, m + 2) - 2(m + 1)(2m + 1)\,\mu\,.\,\mathrm{U}(n, m + 1)\\ + (n - m)(n + m + 1)\,\mathrm{U}(n, m) = 0;\end{aligned}$$

referring to the formulæ (40), (50), we see that, as special cases of this result,

$$\left.\begin{aligned}\mathrm{P}_n^{m+2}(\mu) + 2(m+1)\frac{\mu}{(\mu^2-1)^{\frac{1}{2}}}\,\mathrm{P}_n^{m+1}(\mu) - (n-m)(n+m+1)\,\mathrm{P}_n^{m}(\mu) = 0\\ \mathrm{Q}_n^{m+2}(\mu) + 2(m+1)\frac{\mu}{(\mu^2-1)^{\frac{1}{2}}}\,\mathrm{Q}_n^{m+1}(\mu) - (n-m)(n+m+1)\,\mathrm{Q}_n^{m}(\mu) = 0\end{aligned}\right\}\ (126),$$

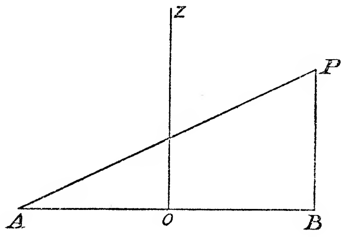
the formulæ (125), (126), are well known for the case in which m and n are real integers.

If $\mu = \cos\theta$, then introducing the modification of Art. 17 into the symbols P_n^m, Q_n^m, we have

$$\left.\begin{aligned}\mathrm{P}_n^{m+2}(\cos\theta) - 2(m+1)\cot\theta\,.\,\mathrm{P}_n^{m+1}(\cos\theta) + (n-m)(n+m+1)\mathrm{P}_n^{m}(\cos\theta) = 0\\ \mathrm{Q}_n^{m+2}(\cos\theta) - 2(m+1)\cot\theta\,.\,\mathrm{Q}_n^{m+1}(\cos\theta) + (n-m)(n+m+1)\mathrm{Q}_n^{m}(\cos\theta) = 0\end{aligned}\right\}\ (127).$$

Toroidal Functions.

56. If A, B are points at the extremities of a diameter of a fixed circle, and the coordinates of any point P in a plane through AB perpendicular to the plane of the circle, are denoted by σ, θ, ϕ, where $\sigma = \log\frac{\mathrm{AP}}{\mathrm{BP}}$, $\theta = \angle\,\mathrm{APB}$, and ϕ is the angle the plane APB makes with a fixed plane through the axis $0z$ which bisects AB and is perpendicular to the plane of the circle, it is known*



that the normal functions requisite for the solution of potential problems connected with the anchor ring are

$$\mathrm{P}_{n-\frac{1}{2}}^{m}(\cosh\sigma)\overset{\cos}{\underset{\sin}{\cdot}}\,n\theta\,\overset{\cos}{\underset{\sin}{\cdot}}\,m\phi,$$

$$\mathrm{Q}_{n-\frac{1}{2}}^{m}(\cosh\sigma)\overset{\cos}{\underset{\sin}{\cdot}}\,n\theta\,\overset{\cos}{\underset{\sin}{\cdot}}\,m\phi.$$

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* See C. NEUMANN'S 'Theorie der Elektricitäts- und Wärme-Vertheilung in einem Ringe,' Halle, 1864. W. M. HICKS, "Toroidal Functions," 'Phil. Trans.,' 1879. A. B. BASSET, "On Toroidal Functions," 'American Journal of Mathematics,' vol. 15. W. D. NIVEN, "On the Ring Functions," 'Proc. Lond. Math. Soc.,' vol. 24.

The functions $P_{n-\frac{1}{2}}^m(\cosh\sigma)$, $Q_{n-\frac{1}{2}}^m(\cosh\sigma)$, where m and n are positive integers, are consequently called toroidal functions. Various expressions for these functions may be found, as particular cases of the various definite integral expressions which have been given above for $P_n^m(\mu)$, $Q_n^m(\mu)$.

We find from (113)

$$P_{n-\frac{1}{2}}(\cosh\sigma) = \frac{2}{\pi}\int_0^\sigma \frac{\cosh n\phi}{\sqrt{(2\cosh\sigma - 2\cosh\phi)}}$$

Also from (81), (82),

$$P_{n-\frac{1}{2}}^m(\cosh\sigma) = \frac{1}{\pi}\cdot\frac{\Pi(n+m-\frac{1}{2})}{\Pi(n-\frac{1}{2})}\int_0^\pi (\cosh\sigma + \sinh\sigma\cos\phi)^{n-\frac{1}{2}}\cos m\phi\, d\phi,$$

$$= \frac{(-1)^m}{\pi}\cdot\frac{\Pi(n-\frac{1}{2})}{\Pi(n-m-\frac{1}{2})}\int_0^\pi \frac{\cos m\phi}{(\cosh\sigma + \sinh\sigma\cos\phi)^{m-\frac{1}{2}}}\, d\phi.$$

From (68), (69)

$$P_{n-\frac{1}{2}}^m(\cosh\sigma)$$

$$= \frac{\Pi(n+m-\frac{1}{2})}{\Pi(n-m-\frac{1}{2})}\cdot\frac{1}{2^m\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\sinh^m\sigma\int_0^\pi (\cosh\sigma + \sinh\sigma\cos\phi)^{n-m-\frac{1}{2}}\sin^{2m}\phi\, d\phi,$$

$$= \frac{\Pi(n+m-\frac{1}{2})}{\Pi(n-m-\frac{1}{2})}\,\frac{1}{2^m\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\sinh^m\sigma\int_0^\pi \frac{\sin^{2m}\phi}{(\cosh\sigma + \sinh\sigma\cos\phi)^{n+m+\frac{1}{2}}}\, d\phi.$$

Again from (92), we find

$$Q_{n-\frac{1}{2}}^m(\cosh\sigma) = (-1)^m\frac{\Pi(n+m-\frac{1}{2})}{\Pi(n-\frac{1}{2})}\int_0^{\log\coth\frac{\sigma}{2}} (\cosh\sigma - \sinh\sigma\cosh w)^{n-\frac{1}{2}}\cosh mw\, dw,$$

and from (122),

$$Q_{n-\frac{1}{2}}^m(\cosh\sigma) = (-1)^m.\frac{2^m\Pi(m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{\pi}\sinh^m\sigma\int_0^\pi \frac{\cos n\phi}{(2\cosh\sigma - 2\cos\phi)^{m+\frac{1}{2}}}\, d\phi.$$

In the case in which the real part of $n - m + \frac{1}{2}$ is positive, we find from (90) and (91),

$$Q_{n-\frac{1}{2}}^m(\cosh\sigma)$$

$$= (-1)^m.2^m.\frac{\Pi(n+m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{\Pi(n-m-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\sinh^m\sigma\int_0^{\log\coth\frac{\sigma}{2}} (\cosh\sigma - \sinh\sigma\cosh w)^{n-m-\frac{1}{2}}\sinh^{2m}w\, dw,$$

$$= (-1)^m\frac{\Pi(n-\frac{1}{2})}{\Pi(n-m-\frac{1}{2})}\int_0^\infty \frac{\cosh mw}{(\cosh\sigma + \sinh\sigma\cosh w)^{n+\frac{1}{2}}}\, dw.$$

57. From (125), (126), we find, on writing $n - \frac{1}{2}$ for n, the relations

$$2n \cosh \sigma . \mathrm{P}_{n-\frac{1}{2}}^{m}(\cosh \sigma) - (n - m + \tfrac{1}{2}) \mathrm{P}_{n+\frac{1}{2}}^{m}(\cosh \sigma) - (n + m - \tfrac{1}{2}) \mathrm{P}_{n-\frac{3}{2}}^{m}(\cosh \sigma) = 0,$$

with a similar relation for the Q functions, and

$$\mathrm{P}_{n-\frac{1}{2}}^{m+2}(\cosh \sigma) + 2(m+1) \coth \sigma . \mathrm{P}_{n-\frac{1}{2}}^{m+1}(\cosh \sigma) \\ - (n - m - \tfrac{1}{2})(n + m + \tfrac{1}{2}) \mathrm{P}_{n-\frac{1}{2}}^{m}(\cosh \sigma) = 0,$$

with a similar relation for the Q functions. Formulæ similar to these have been employed by HICKS to calculate the functions successively.

58. It is important to have series for $\mathrm{P}_{n-\frac{1}{2}}^{m}(\cosh \sigma)$, $\mathrm{Q}_{n-\frac{1}{2}}^{m}(\cosh \sigma)$ in powers of $e^{-\sigma}$, so that the values of the functions may be calculated approximately for considerable values of σ. The required series for $\mathrm{Q}_{n-\frac{1}{2}}^{m}(\cosh \sigma)$ is given at once by (35); we thus have

$$\mathrm{Q}_{n-\frac{1}{2}}^{m}(\cosh \sigma) \\ = (-1)^m 2^m \frac{\Pi(n + m - \frac{1}{2}) \Pi(-\frac{1}{2})}{\Pi(n)} \sinh^m \sigma . e^{-(n+m+\frac{1}{2})\sigma} \mathrm{F}(m + \tfrac{1}{2}, n + m + \tfrac{1}{2}, n + 1, e^{-2\sigma}),$$

and in particular

$$\mathrm{Q}_{n-\frac{1}{2}}(\cosh \sigma) = \frac{\Pi(n - \frac{1}{2}) \Pi(-\frac{1}{2})}{\Pi(n)} e^{-(n+\frac{1}{2})\sigma} \mathrm{F}(\tfrac{1}{2}, n + \tfrac{1}{2}, n + 1, e^{-2\sigma}).$$

This is the expansion in powers of $e^{-\sigma}$, of the elliptic integral to which

$$\int_0^\infty \frac{dw}{\sqrt{(\cosh \sigma + \sinh \sigma \cosh w)}}$$

is reduced by means of the substitution $\cosh \sigma + \sinh \sigma \cosh w = \operatorname{cosec}^2 \theta . e^{\sigma}$.

The corresponding series for $\mathrm{P}_{n-\frac{1}{2}}^{m}(\cosh \sigma)$ must be obtained from (36), which requires, however, in this case modification. We observe that in the formula

$$\mathrm{P}_{n}^{m}(\cosh \sigma) \\ = 2^m \frac{\sin(n+m)\pi}{\cos n\pi} \cdot \frac{\Pi(n+m)}{\Pi(n+\frac{1}{2}) \Pi(-\frac{1}{2})} \sinh^m \sigma\, e^{-(n+m+1)\sigma} \mathrm{F}(m+\tfrac{1}{2}, n+m+1, n+\tfrac{3}{2}, e^{-2\sigma}) \\ + 2^m \frac{\Pi(n - \frac{1}{2})}{\Pi(n - m) \Pi(-\frac{1}{2})} \sinh^m \sigma . e^{(n-m)\sigma} \mathrm{F}(m + \tfrac{1}{2}, m - n, \tfrac{1}{2} - n, e^{-2\sigma})$$

when $n - \frac{1}{2}$ is a positive integer p_0; the second series has after a finite number of terms, infinite coefficients, moreover the coefficient $\sec n\pi$ of the first series is infinite.

The expression for $\mathrm{P}_{n}^{m}(\cosh \sigma)$, gives us, therefore, first a finite series

$$2^m \frac{\Pi(p_0)}{\Pi(p_0+\frac{1}{2}-m)\,\Pi(-\frac{1}{2})} \sinh^m\sigma \,.\, e^{(p_0+\frac{1}{2}-m)\sigma} \left\{ 1 + \frac{(\frac{1}{2}+m)(p_0+\frac{1}{2}-m)}{1\,.\,p_0} e^{-2\sigma} \right.$$

$$+ \frac{(\frac{1}{2}+m)(\frac{3}{2}+m)(p_0-m+\frac{1}{2})(p_0-m-\frac{1}{2})}{1\,.\,2\,.\,p_0\,.\,p_0-1} e^{-4\sigma} + \dots$$

$$\left. + \frac{(\frac{1}{2}+m)\dots(\frac{1}{2}+m+p_0-1)(p_0-m+\frac{1}{2})\dots(-m+\frac{1}{2})}{1\,.\,2\dots p_0\,.\,p_0(p_0-1)\dots 1} \,.\, e^{-2p_0\sigma} \right\},$$

which we shall denote by S_1; and second the undetermined form

$$2^m \frac{\sin(p+\frac{1}{2}+m)\pi}{\cos(p+\frac{1}{2})\pi} \,.\, \frac{\Pi(p+\frac{1}{2}+m)}{\Pi(p+1)\,\Pi(-\frac{1}{2})} \sinh^m\sigma \,.\, e^{-(p+m+\frac{3}{2})\sigma}\, \mathrm{F}(m+\tfrac{1}{2}, p+\tfrac{3}{2}+m, p+2, e^{-2\sigma})$$

$$+ 2^m \frac{\Pi(p)}{\Pi(p+\frac{1}{2}-m)\,\Pi(-\frac{1}{2})} \, \frac{(\frac{1}{2}+m)\dots(\frac{1}{2}+m+p_0)(p-m+\frac{1}{2})\dots(p-m+\frac{1}{2}-p_0)}{1\,.\,2\dots(p_0+1)\,.\,p(p-1)\dots(p-p_0)} \,.\, \sinh^m\sigma$$

$$e^{(p+\frac{1}{2}-m-2p_0-2)\sigma} \left\{ 1 + \frac{\frac{1}{2}+m+p_0+1\,.\,p-m+\frac{1}{2}-p_0-1}{p_0+2\,.\,p-p_0-1} e^{-2\sigma} + \dots \right\}$$

where in the limit, $p = p_0$.

The numerical coefficient of the second series is equal to

$$2^m \,.\, \frac{\Pi(p)}{\Pi(p+\frac{1}{2}-m)\,\Pi(-\frac{1}{2})} \,.\, \frac{\Pi(p_0+m+\frac{1}{2})}{\Pi(m-\frac{1}{2})} \,.\, \frac{\Pi(p-m+\frac{1}{2})}{\Pi(p-p_0-m-\frac{1}{2})} \,.\, \frac{\Pi(p-p_0-1)}{\Pi(p)} \,.\, \frac{1}{\Pi(p_0+1)},$$

which is equal to

$$\frac{2^m}{\Pi(p_0+1)} \,.\, \frac{\Pi(p_0+m+\frac{1}{2})}{\Pi(m-\frac{1}{2})\,\Pi(-\frac{1}{2})} \,.\, \frac{\Pi(p_0-p+m-\frac{1}{2})}{\Pi(p_0-p)} \,.\, \frac{\sin(p_0-p+m+\frac{1}{2})\pi}{\sin(p_0-p+1)\pi} \,.$$

Now the limiting value of the ratio

$$\frac{\sin(p+\frac{1}{2}+m)\pi}{\cos(p+\frac{1}{2})\pi} \Big/ \frac{\sin(p_0-p+m+\frac{1}{2})\pi}{\sin(p_0-p+1)\pi}$$

when $p = p_0$, is easily seen to be -1, thus the coefficients of the two series are equal and opposite infinities.

Evaluating the indeterminate form according to the known rule, we obtain first an expression, which we shall denote by S_2; this is

$$S_2 = 2^m \frac{\Pi(p_0+m+\frac{1}{2})}{\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})\,\Pi(p_0+1)} \,.\, \sinh^m\sigma \,.\, \mathrm{L}_{p=p_0} \frac{p-p_0}{\sin(p-p_0)}$$

$$\mathrm{L}_{p=p_0} \frac{d}{dp} \left[\frac{\Pi(p_0-p+m-\frac{1}{2})}{\Pi(p_0-p)} \sin(p_0-p+m+\tfrac{1}{2})\pi \,.\, e^{-(m+\frac{3}{2})\sigma+(p-2p_0)\sigma} \right.$$

$$\left. \left\{ 1 + \frac{\frac{3}{2}+m+p_0\,.\,p-p_0-m-\frac{1}{2}}{p_0+2\,.\,p-p_0-1} \,.\, e^{-2\sigma} + \dots \right\} \right];$$

we also obtain the expression

$$S_3 = -2^m(-1)^{p_0}\sinh^m\sigma \,.\, \frac{d}{dp_0}\left\{\frac{\Pi(p_0+\tfrac{1}{2}+m)}{\Pi(p_0+1)\,\Pi(-\tfrac{1}{2})}\sin(p_0+\tfrac{1}{2}+m)\,\pi\right.$$
$$\left. e^{-(p_0+m+\frac{3}{2})\sigma}\,F(m+\tfrac{1}{2}, p_0+\tfrac{3}{2}+m, p_0+2, e^{-2\sigma})\right\},$$

since $\cos(p+\frac{1}{2})\,\pi = -(-1)^{p_0}(p-p_0)$, in the limit.

We have now, on the whole, picking out the terms in S_2, S_3, obtained by differentiating the exponential function

(1) The finite series S_1,

(2) $$2^{m+1}\,.\,\sin(m+\tfrac{1}{2})\,\pi\,.\,\frac{\Pi(p_0+m+\frac{1}{2})}{\Pi(p_0+1)\,\Pi(-\frac{1}{2})}\,\sigma\,.\,\sinh^m\sigma\,.\,e^{-(p_0+m+\frac{3}{2})\sigma}$$
$$F(m+p_0+\tfrac{3}{2}, m+\tfrac{1}{2}, p_0+2, e^{-2\sigma}),$$

(3) $$-2^{m+1}\frac{\Pi(p_0+m+\frac{1}{2})}{\Pi(p_0+1)\,\Pi(-\frac{1}{2})}\cos(m+\tfrac{1}{2})\,\pi\,.\,\sinh^m\sigma\,.\,e^{-(p_0+m+\frac{3}{2})\sigma}$$
$$F(m+\tfrac{1}{2}, m+p_0+\tfrac{3}{2}, p_0+2, e^{-2\sigma}),$$

$$+\frac{2^m\,.\,\Pi(p_0+m+\frac{1}{2})}{\Pi(-\frac{1}{2})\,\Pi(p_0+1)}\sin(m+\tfrac{1}{2})\,\pi\left\{\frac{\Pi'(0)}{\Pi(0)}-\frac{\Pi'(m-\frac{1}{2})}{\Pi(m-\frac{1}{2})}-\frac{\Pi'(p_0+m+\frac{1}{2})}{\Pi(p_0+m+\frac{1}{2})}\right.$$
$$\left.+\frac{\Pi'(p_0+1)}{\Pi(p^0+1)}\right\}\sinh^m\sigma\,.\,e^{-(p_0+m+\frac{3}{2})\sigma}\,F(m+\tfrac{1}{2}, m+p_0+\tfrac{3}{2}, p_0+2, e^{-2\sigma}),$$

in the case of the ordinary ring functions (m integral), the first term vanishes on account of the factor $\cos(m+\frac{1}{2})\,\pi$.

(4) $$\frac{2^m\sin(m+\frac{1}{2})\,\pi}{\Pi(-\frac{1}{2})}\sinh^m\sigma\,.\,e^{-(p_0+m+\frac{3}{2})\sigma}.$$

$$\sum_{s=1}^{s=\infty}\frac{\Pi_0(p_0+m+s+\frac{1}{2})}{\Pi(p_0+s+1)}\,\frac{\Pi(m+s-\frac{1}{2})}{\Pi(m-\frac{1}{2})\,\Pi(s)}\left\{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{s}\right.$$
$$-\frac{1}{m+\frac{1}{2}}-\frac{1}{m+\frac{3}{2}}-\ldots-\frac{1}{m+s-\frac{1}{2}}+\frac{1}{p_0+2}+\ldots+\frac{1}{p_0+s+1}$$
$$\left.-\frac{1}{p_0+m+\frac{3}{2}}-\ldots-\frac{1}{p_0+m+s+\frac{1}{2}}\right\}e^{-2s\sigma}.$$

Confining ourselves to the case in which m is integral, we can simplify the expression in (2); we have

$$\frac{\Pi'(p_0+1)}{\Pi(p_0+1)}=\frac{1}{p+1}+\frac{1}{p}+\ldots+\frac{1}{1}+\frac{\Pi'(0)}{\Pi(0)}$$

$$\frac{\Pi'(m-\frac{1}{2})}{\Pi(m-\frac{1}{2})}=\frac{1}{m-\frac{1}{2}}+\frac{1}{m-\frac{3}{2}}+\ldots+\frac{1}{\frac{1}{2}}+\frac{\Pi'(-\frac{1}{2})}{\Pi(-\frac{1}{2})}$$

$$\frac{\Pi'(m+p_0+\frac{1}{2})}{\Pi(m+p_0+\frac{1}{2})}=\frac{1}{m+p_0+\frac{1}{2}}+\ldots+\frac{1}{\frac{1}{2}}+\frac{\Pi'(-\frac{1}{2})}{\Pi(-\frac{1}{2})},$$

hence

$$\frac{\Pi'(0)}{\Pi(0)} - \frac{\Pi'(m-\frac{1}{2})}{\Pi(m-\frac{1}{2})} - \frac{\Pi'(p_0+m+\frac{1}{2})}{\Pi(p_0+m+\frac{1}{2})} + \frac{\Pi'(p_0+1)}{\Pi(p_0+1)}$$

$$= 2\left\{\frac{\Pi'(0)}{\Pi(0)} - \frac{\Pi'(-\frac{1}{2})}{\Pi(-\frac{1}{2})}\right\} + \left[\frac{1}{1} + \ldots + \frac{1}{p_0+1} - \left(\frac{1}{\frac{1}{2}} + \ldots + \frac{1}{m-\frac{1}{2}}\right)\right.$$
$$\left. - \left(\frac{1}{\frac{1}{2}} + \ldots + \frac{1}{p_0+m+\frac{1}{2}}\right)\right].$$

Now use the known theorem $\Pi(x-1)\,\Pi(x-\frac{1}{2}) = \sqrt{2\pi}\,.\,2^{\frac{1}{2}-2x}\,.\,\Pi(2x-1)$; on taking logarithms and differentiating, and then putting $x=\frac{1}{2}$, we find

$$\frac{\Pi'(0)}{\Pi(0)} - \frac{\Pi'(-\frac{1}{2})}{\Pi(-\frac{1}{2})} = \log_e 4.$$

Taking (2), (3), and (4) together, we now have the expression

$$(-1)^m 2^{m+1}\frac{\Pi(p_0+m+\frac{1}{2})}{\Pi(p_0+1)\Pi(-\frac{1}{2})}\log(4e^\sigma).\sinh^m\sigma.e^{-(p_0+m+\frac{3}{2})\sigma}\mathrm{F}(m+p_0+\tfrac{3}{2}, m+\tfrac{1}{2}, p_0+2, e^{-2\sigma})$$
$$+(-1)^m 2^m.\frac{\sinh^m\sigma\,.\,e^{-(p_0+m+\frac{3}{2})\sigma}}{\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\sum_{s=0}^{s=\infty}(u_{p_0+s+1} + u_s + v_{m+s-\frac{1}{2}} - v_{p_0+m+s+\frac{1}{2}})$$
$$\frac{\Pi(p_0+m+s+\frac{1}{2})\,\Pi(m+s-\frac{1}{2})}{\Pi(s)}e^{-2s\sigma},$$

where

$$u_r \text{ denotes the series } \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{r}$$

and

$$v_{r+\frac{1}{2}} \text{ denotes the series } \frac{1}{\frac{1}{2}} + \frac{1}{\frac{3}{2}} + \ldots + \frac{1}{r+\frac{1}{2}}.$$

On changing p_0 into $n-1$, we now have the complete expression for the ring function $\mathrm{P}_{n-\frac{1}{2}}{}^m(\cosh\sigma)$, ($m$ integral),

$$\mathrm{P}_{n-\frac{1}{2}}{}^m(\cosh\sigma)$$
$$= \frac{2^m\Pi(n-1)}{\Pi(n-m-\frac{1}{2})\,\Pi(-\frac{1}{2})}\sinh^m\sigma\,.\,e^{-(n-m-\frac{1}{2})\sigma}\left[1 + \frac{(\frac{1}{2}+m)(n-\frac{1}{2}-m)}{1\,.\,n-1}e^{-2\sigma} + \ldots\right.$$
$$\left. + \frac{(\frac{1}{2}+m)\ldots(\frac{1}{2}+m+n-2)\,(n-m-\frac{1}{2})\ldots(-m+\frac{1}{2})}{1\,.\,2\ldots n-1\,.\,n-1\,.\,n-2\ldots1}e^{-2(n-1)\sigma}\right]$$
$$+(-1)^m 2^{m+1}.\frac{\Pi(m+n-\frac{1}{2})}{\Pi(n)\,\Pi(-\frac{1}{2})}\log(4e^\sigma)\sinh^m\sigma\,.\,e^{-(n+m+\frac{1}{2})\sigma}\mathrm{F}(m+n+\tfrac{1}{2}, m+\tfrac{1}{2}, n+1, e^{-2\sigma})$$
$$+(-1)^m 2^m.\frac{\sinh^m\sigma\,.\,e^{-(n+m+\frac{1}{2})\sigma}}{\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\sum_{s=0}^{s=\infty}(u_{s+n} + u_s - v_{m+s-\frac{1}{2}} - v_{m+n+s-\frac{1}{2}})$$
$$\frac{\Pi(m+n+s-\frac{1}{2})\,\Pi(m+s-\frac{1}{2})}{\Pi(s)}e^{-2s\sigma}.$$

The particular case $m = 0$, gives as the expression for the zonal function,

$$
\begin{aligned}
&\mathrm{P}_{n-\frac{1}{2}}(\cosh\sigma) \\
&= \frac{\Pi(n-1)}{\Pi(n-\frac{1}{2})\Pi(-\frac{1}{2})} e^{-(n-\frac{1}{2})\sigma}\left[1 + \frac{\frac{1}{2}.n-\frac{1}{2}}{1.n-1} e^{-2\sigma} + \ldots + \frac{\frac{1}{2}.\frac{3}{2}\ldots(n-\frac{3}{2})(n-\frac{1}{2})(n-\frac{3}{2})\ldots\frac{1}{2}}{1.2\ldots n-1.n-1\ldots 1} e^{-2(n-1)\sigma}\right] \\
&\quad + \frac{2\,\Pi(n-\frac{1}{2})}{\Pi(n)\,\Pi(-\frac{1}{2})} \log(4e^{\sigma})\, e^{-(n+\frac{1}{2})\sigma}\,\mathrm{F}(n+\tfrac{1}{2},\ \tfrac{1}{2},\ n+1,\ e^{-2\sigma}) \\
&\quad + \frac{1}{\pi} e^{-(n+\frac{1}{2})\sigma} \sum_{0}^{\infty} (u_{n+s} + u_s - v_{s-\frac{1}{2}} - v_{n+s-\frac{1}{2}}) \frac{\Pi(n+s-\frac{1}{2})\,\Pi(s-\frac{1}{2})}{\Pi(s)} e^{-2s\sigma}.
\end{aligned}
$$

This particular case has been obtained by other methods by **Basset**, and by W. D. **Niven**.

The case in which m is fractional has really been included in the above investigation; the simplification of the coefficients in the expression (2) does not apply to the general case.

Mehler's *Functions for the Cone.*

59. The normal potential functions for problems in which the boundaries are coaxal circular cones* are spherical harmonics of complex degree $-\frac{1}{2} + p\iota$; it is therefore desirable to consider the forms which the functions $\mathrm{P}_n(\cos\theta)$, $\mathrm{Q}_n(\cos\theta)$ take when n is of this form; $\mathrm{P}_{-\frac{1}{2}+p\iota}(\cos\theta)$ will be denoted by $\mathrm{K}_p(\cos\theta)$.

We find from (103), (110), (111),

$$
\begin{aligned}
\mathrm{K}_p(\cos\theta) &= \frac{2}{\pi}\int_0^{\theta} \frac{\cosh pu}{\sqrt{2\cos u - 2\cos\theta}}\,du, \\
&= \frac{2}{\pi}\cosh p\pi \int_0^{\infty} \frac{\cos pv}{\sqrt{2\cosh v + 2\cos\theta}}\,dv;
\end{aligned}
$$

these formulæ have been proved by other methods by **Mehler** and by **Heine**.†

From (103), we obtain the new formula

$$
\mathrm{K}_p^{m}(\cos\theta) = (-1)^m \frac{2}{2^m\,\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})} \frac{\Pi(n+m)}{\Pi(n-m)} \sin^{-m}\theta \int_0^{\theta} \frac{\cosh pu}{(2\cos u - 2\cos\theta)^{\frac{1}{2}-m}}
$$

where m is any positive quantity.

From the above formulæ, we see that $\mathrm{P}_{-\frac{1}{2}+p\iota}(\cos\theta) = \mathrm{P}_{-\frac{1}{2}-p\iota}(\cos\theta)$.

From (117), we have,

---

* See **Mehler**'s paper in **Crelle**'s 'Journal,' vol. 68.

† See 'Kugelfunctionen,' vol. 2, p. 221.

$$Q_{-\frac{1}{2}+p\iota}(\cos\theta) - \frac{\iota\pi}{2}P_{-\frac{1}{2}+p\iota}(\cos\theta) = Q_{-\frac{1}{2}+p\iota}(\cos\theta + 0\,.\,\iota)$$

$$= \int_0^\infty \frac{\cos pu}{\sqrt{2\cosh u - 2\cos\theta}}\,du - \iota e^{+p\pi}\int_0^\infty \frac{\cos pu}{\sqrt{2\cosh u + 2\cos\theta}}\,du,$$

hence, changing p into $-p$, and adding the two equations, we find

$$Q_{-\frac{1}{2}+p\iota}(\cos\theta) + Q_{-\frac{1}{2}-p\iota}(\cos\theta) - \iota\pi P_{-\frac{1}{2}+p\iota}(\cos\theta)$$

$$= 2\int_0^\infty \frac{\cos pu}{\sqrt{2\cosh u - 2\cos\theta}}\,du - 2\iota\cosh p\pi\int_0^\infty \frac{\cos pu}{\sqrt{2\cosh u + 2\cos\theta}}\,du,$$

hence

$$\frac{\cosh p\pi}{\pi}\{Q_{-\frac{1}{2}+p\iota}(\cos\theta) + Q_{-\frac{1}{2}-p\iota}(\cos\theta)\} = \frac{2\cosh p\pi}{\pi}\int_0^\infty \frac{\cos pu}{\sqrt{2\cosh u - 2\cos\theta}}$$

$$= K_p(-\cos\theta).$$

Thus we can use $K_p(\cos\theta)$, $K_p(-\cos\theta)$, as the two independent functions.

It thus appears that the expressions given by MEHLER and HEINE for $K_p(\cos\theta)$, $K_p(-\cos\theta)$, are particular cases of the general formulæ we have obtained above.

Potential Functions for the Bowl.

60. It has been shown by MEHLER that for potential problems in which the boundaries are spherical bowls with a common circular rim, the functions $K_p(\mu)$ can be used, μ being in this case real and greater than unity, say $\mu = \cosh\psi$.

We find from (111), that

$$P_{-\frac{1}{2}+p\iota}(\cosh\psi) = K_p(\cosh\psi) = \frac{2}{\pi}\cosh p\pi\int_0^\infty \frac{\cos pu}{\sqrt{2\cosh u + 2\cosh\psi}}\,du,$$

and from (113), we find

$$K_p(\cosh\psi) = \frac{2}{\pi}\int_0^\psi \frac{\cos pv}{\sqrt{2\cosh\psi - 2\cosh v}}\,dv,$$

also from (115),

$$K_p(\cosh\psi) = \frac{2}{\pi}\coth p\pi\int_\psi^\infty \frac{\sin pw}{\sqrt{2\cosh w - 2\cosh\psi}}\,dw,$$

these formulæ are all proved by HEINE[*] by other methods.

From (112) we have

$$K_p^m(\cosh\psi) = \frac{2(-1)^m}{2^m\Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})}\frac{\Pi(n+m)}{\Pi(n-m)}\sinh^{-m}\psi\int_0^\psi \frac{\cos pu}{(2\cosh\psi - 2\cosh u)^{\frac{1}{2}-m}}\,du$$

---

[*] See 'Kugelfunctionen,' vol. 2, p. 220.

From (118), we have

$$Q_{-\frac{1}{2}+p\iota}(\cosh\psi) = \int_{\psi}^{\infty} \frac{e^{-p\iota u}}{\sqrt{2\cosh u - 2\cosh\psi}}\,du.$$

Hence

$$Q_{-\frac{1}{2}+p\iota}(\cosh\psi) + Q^{-\frac{1}{2}-p\iota}(\cosh\psi) = 2\int_{\psi}^{\infty} \frac{\cos pu}{\sqrt{2\cosh u - 2\cosh\psi}}\,du,$$

hence defining $K^{p}(-\cosh\psi)$ by means of the formula

$$K_{p}(-\cosh\psi) = \frac{2}{\pi}\cosh\, p\pi \int_{0}^{\infty} \frac{\cos pu}{\sqrt{2\cosh u - 2\cosh\psi}}\,du,$$

we have

$$\frac{\cos p\pi}{\pi}\left\{Q_{-\frac{1}{2}+p\iota}(\cosh\psi) + Q_{-\frac{1}{2}-p\iota}(\cosh\psi)\right\} = K_{p}(-\operatorname{ccsh}\psi).$$

It thus appears that the known expressions for these functions are immediately derivable from the general formulæ obtained in the present memoir.